HIGHER HOCHSCHILD COHOMOLOGY, BRANE TOPOLOGY AND CENTRALIZERS OF E_n-ALGEBRA MAPS

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ABSTRACT. We obtain an E_{n+1}-algebra model on $C_{\bullet+m}(Map(S^n, M))$, the shifted integral chains on the mapping space of the n-sphere into an m-dimensional orientable closed manifold M. Our main tool is factorization homology and higher Hochschild (co)chains and we discuss some other applications of these tools of independent interest. We construct and use E_∞-Poincaré duality to identify the higher Hochschild cochains, modeled over the n-sphere, with the chains on the above mapping space, and then relate the Hochschild cochains to the deformation complex of the E_∞-algebra $C^*(M)$, thought of as an E_n-algebra. We then invoke (and prove) the higher Deligne conjecture to furnish the cotangent complex, and all that is naturally equivalent to it, with an E_{n+1}-algebra structure and further prove that this construction recovers the sphere product. In fact, our approach to Deligne conjecture is based on an explicit description of the E_n-centralizers of a map of E_∞-algebras $f : A \to B$ by relating it to the algebraic structure on Hochschild cochains modeled over spheres, which is of independent interest. The latter also applies to iterated Bar construction for E_∞-algebras together with their E_n-coalgebra and E_∞-algebra structure. In particular, we give a higher Hochschild chain model of the natural E_n-algebra structure of the chains of the iterated loop space $C_*(\Omega^n Y)$. Furthermore, for general E_n-algebras, we apply factorization algebras to give a construction of the centralizers of any map of E_n-algebras, solutions of higher Deligne conjecture and to discuss several features of the iterated bar construction for E_n-algebras.

CONTENTS

1. Introduction

The main objective of this paper is the study of the algebraic structure of the integral chains on the mapping space of the n-sphere into an orientable m-dimensional manifold M. Our main tools are factorization algebras and factorization homology. We also demonstrate the efficiency of these tools to study *iterated Bar construction* and *iterated loop spaces* and to describe the structure of the *centralizer* of E_n-algebra maps as well as a solution to the higher Deligne conjectures. These applications are the core of sections 7, 8 and 9 which deal with generalizations of the Hochschild cohomology to E_n and E_∞-algebras and their algebraic structures, using the concept of factorization homology.

The algebraic structure of the chains on the mapping spaces of spheres into a manifold has drawn considerable interest, following the work of Chas-Sullivan [CS] on the free loop space. The homology of the free loop space $LM = Map(S^1, M)$, shifted by the dimension of M, has the structure of a BV-algebra, and in particular of a Gerstenhaber algebra, or a 1-Poisson algebra, that is of a (graded) commutative algebra endowed with a degree 1 Lie bracket satisfying the Leibniz rule. This structure is now known to be part of a 2-dimensional homological conformal field theory [Go, BGNX]. The BV-algebra structure lies in the genus 0 part of this topological conformal field theory. *Higher string topology*, also referred to as *Brane topology*, is a generalization of string topology in which the circle is replaced by the n-dimensional sphere. Sullivan and Voronov (see [CV]) have stated[1] that the (shifted) homology of the mapping sphere $Map(S^n, M)$ has the structure of a BV$_n$-algebra and in particular of a n-Poisson algebra (or n-braid algebra in the terminology of [KM]). The latter structure is the analogue of a Gerstenhaber algebra in which the Lie bracket is of degree n. A BV$_n$-algebra is an algebra over the homology of the operad of framed n-dimensional little disks, *i.e.*, the framed E_n-operad; while an n-Poisson algebra is an algebra over the homology of the E_n-operad, for instance see [CV, SW]. Sullivan-Voronov work leads to the following question: is it possible to lift the n-Poisson algebra structure on the homology of

[1] also see [CV, BGNX, C] for explicit construction of the underlying graded commutative multiplication, called the sphere product

$Map(S^n, M)$ to a structure of E_n-algebras on the chains of $Map(S^n, M)$? For an n-connected closed and oriented manifold M, we give a positive answer to this conjecture:

Theorem 8.1. *Let M be a n-connected Poincaré duality space whose homology groups are projective k-modules. The shifted chain complex $C_{*+\dim(M)}(Map(S^n, M))$ has a natural E_{n+1}-algebra structure which induces the Sullivan-Voronov sphere product in homology*

$$H_p\big(Map(S^n, M)\big) \otimes H_q\big(Map(S^n, M)\big) \to H_{p+q-\dim(M)}\big(Map(S^n, M)\big),$$

when M is an oriented closed manifold.

This E_{n+1}-algebra structure can be seen an higher dimensional analogue of the genus 0 part of a topological conformal field theory.

Our approach is based on an *algebraic model* of the chains on the mapping spaces generalizing *Hochschild cochains* whose use proved to be a fruitful model for string topology operations. Hochschild cohomology groups of an algebra A with value in a bimodule N are defined as

$$HH^n(A, N) \cong H^n\big(\mathbb{R}Hom_{A\otimes A^{op}}(A, N)\big) \cong Ext^n_{A\otimes A^{op}}(A, N).$$

while the Hochschild homology groups $HH_\bullet(A, N) \cong Tor_n^{A\otimes A^{op}}(A, N)$ are defined similarly by derived tensor products. These (co)homology groups are given by standard (co)chain complexes, for instance see [Ge, L]. The Hochschild cohomology of any associative (or E_1) algebra has a natural Gerstenhaber algebra structure and further, by the (solutions to the) Deligne conjecture, the latter is induced by an E_2-algebra structure on the Hochschild cochains. Hochschild (co)chains are a model for cochains on the free loop space and string topology. Indeed, there is an isomorphism [CV, FTV]

$$H_\bullet(LM) \cong HH^\bullet(C^*(M), C_*(M)) \cong HH^\bullet(C^*(M), C^*(M))[d] \tag{1}$$

if M is an oriented and simply connected manifold of dimension d which, in characteristic zero is an isomorphism of Gerstenhaber algebras [FT]. Further, Hochschild chains of Calabi-Yau (E_1-)algebras carry a topological conformal field theories structure [Lu4]. The above isomorphisms (1) make use of two ingredients. First, it uses the (dual of) an isomorphism $HH_\bullet(C^*(M), C_*(M)) \cong H_\bullet(LM)$ for any simply connected space M (which can be described in geometric terms by Chen interated integrals when M is a manifold) and, second, it uses a lift of the Poincaré duality quasi-isomorphism $C^*(M) \to C_*(M)[\dim(M)]$ to a bimodule map, when M is further a closed manifold. In this paper, we study and use the generalizations of these two facts for n-dimensional spheres as well as the E_2-algebra structure on Hochschild cochains as we explain below. Combining these three ingredients will give us the desired E_{n+1}-algebra structure on $C_*(Map(S^n, M))$. Our technique should be related to those of Hu [Hu] and Hu-Kriz-Voronov [HKV].

Bimodules over an associative algebra correspond to the *operadic* notion of E_1-modules. There is a notion of E_n-Hochschild cohomology where maps of A-bimodules are replaced by maps of A-E_n-modules for an E_n-algebra A, see [L-HA, F1, Fre]. The Kontsevich-Soibelman generalization of Deligne conjecture, *i.e.*, the higher Deligne conjecture, is that the E_n-Hochschild cohomology of A, denoted $HH_{\mathcal{E}_n}(A, A)$ is an E_{n+1}-algebra. For X a topological space, the cochains $C^*(X)$ are more than simply an associative algebra but are *homotopy commutative*, that is,

carries a functorial structure of E_∞-algebra; in particular of an E_n-algebra for all n. In characteristic zero, one can use CDGAs models for the cochains, but this is not possible when working over integers or a finite field. Nevertheless, for E_∞-algebras, E_n-Hochschild cohomology get extra functoriality (not shared by all E_n-algebras) and actually identifies with higher Hochschild cohomology over spaces, a notion dual to higher Hochschild homology.

The latter theories are the subject of Section 3 and can be expressed in terms of *factorization homology*, also referred to as *topological chiral homology* [L-HA, F1, CG, GTZ2]. Factorization homology is an invariant of *both* (framed) manifolds (and framed embeddings) and E_n-algebras based on (extended) topological field theories. In fact, the factorization homology of E_∞-algebras becomes a homotopy invariant and can be applied to any space (and continuous maps) and not just to framed manifolds. This generalization is precisely computed by *higher Hochschild homology*, introduced by Pirashvili in [P], which can be seen as a kind of *limit* of these ideas when the dimension of the TFT goes to infinity [GTZ2]. Indeed, by Theorem 3.11 below (and [GTZ2, F1, L-HA]) that if X is a manifold and A an E_∞-algebra, then, the factorization homology $\int_X A$ of X with coefficients in A is naturally equivalent to the Hochschild chains $CH_X(A)$ of A over X.

The restriction to E_∞-algebras is not an issue in our case of interest since the cochain complex $C^*(X)$ is indeed an E_∞-algebra. We study $t$he higher Hochschild chains for E_∞-algebras and modules in Section 3.1, which is modeled over spaces in the same way the usual Hochschild (co)chain is modeled on circles. More precisely, this is a rule that assigns to any space X, E_∞-algebra A, and A-module M, a chain complex $CH_X(A, M)$, *functorial in every argument*, such that for $X = S^1$, one recovers the usual Hochschild chains. The functoriality with respect to spaces is a key feature which allows us to derive algebraic operations on the higher Hochschild chain complexes from maps of topological spaces. In particular, for any pointed space X, $CH_X(A)$ is naturally an $A \cong CH_{pt}(A)$-module which allows to define *higher Hochschild cochains* $CH^X(A, M) = Hom_A(CH_X(A), M)$ over any pointed space X by dualizing the Hochschild chains (as A-modules), see Section 3.2. The relationship with E_n-Hochschild cohomology is given by the fact that, for any E_∞-algebra A, the Hochschild cochains over the n-dimensional sphere S^n of A with value in itself coincides with its E_n-Hochschild cohomology.

Proposition 7.11. *Let $f : A \to B$ be a map of E_∞-algebra and let B be endowed with the induced A-E_∞-module structure. Then there is a natural equivalence of E_n-algebras:*

$$HH_{\mathcal{E}_n}(A, B) \cong CH^{S^n}(A, B)$$

Higher Hochschild chains have a good axiomatic characterization (similar to Eilenberg-Steenrod axioms) which formally follows from the fact that E_∞-algebras are tensored over spaces, see Corollary 3.27 in Section 3.3.

The aforementioned relationship between free loop spaces and Hochschild chains generalize to every space taking advantage of its functoriality with respect to maps of spaces. In fact, we prove (in Theorem 4.4) that there is a natural map of E_∞-algebras $CH_Y(C^*(X)) \to C^*(Map(Y, X))$ which is a quasi-isomorphism when X is $\dim(Y)$-connected. This is an E_∞-analogue of our previous result [GTZ] for CDGA's in characteristic zero (where we used generalizations of Chen *iterated integrals*).

In Section 5, we study the algebraic structure of higher Hochschild cochains. We first define, for any A-E_∞-algebra B, the (associative) *wedge product*

$$CH^X(A,B)\otimes CH^Y(A,B)\to CH^{X\vee Y}(A,B)$$

and then we prove that when X is a sphere S^d, the wedge product induces a structure of *E_d-algebra on* $CH^{S^d}(A,B)$, generalizing the usual cup-product in Hochschild cohomology.

Theorem 5.11. *Let A be an E_∞-algebra and B an E_∞-A-algebra. The collection of maps $(pinch_{S^d,k}:\mathcal{C}_d(k)\times S^d \longrightarrow \bigvee_{i=1\dots k} S^d)_{k\geq 1}$ makes $CH^{S^d}(A,B)$ into an E_d-algebra, such that the underlying underlying E_1-structure of $CH^{S^d}(A,B)$ agrees with the one given by the* cup-product*,*

$$\cup_{S^d}:\quad CH^{S^d}(A,B)\otimes CH^{S^d}(A,B)\longrightarrow CH^{S^d\vee S^d}(A,B)\longrightarrow CH^{S^d}(A,B).$$

The CDGA version of this result goes back to the first author note [G].

In Section 7 we reinterpret and generalize the above results, to the E_n-algebras case, in terms of *centralizers*. In fact, using factorization algebras, for any map $f:A\to B$ of E_n-algebras, we put a natural E_n-algebra structure on the E_n-Hochschild cohomology $HH_{\mathcal{E}_n}(A,B)$, see Theorem 7.7. We then show that this structure is precisely the centralizer $\mathfrak{z}(f)$ of f in the sense of Lurie [L-HA].

Proposition 7.18. *Let $f:A\to B$ be an E_n-algebra map and endow $HH_{\mathcal{E}_n}(A,B)$ with the E_n-algebra structure given by Theorem 7.7. Then the $\mathcal{E}_n$-Hochschild cohomology $HH_{\mathcal{E}_n}(A,B)\cong RHom_A^{\mathcal{E}_n}\big(A,B\big)$ is the centralizer $\mathfrak{z}(f)$.*

In particular, it is the solution to a universal problem. As a consequence, using an approach due to Lurie [L-HA], this yields a *solution* to the higher Deligne conjecture. Indeed the naturality of the centralizer with respect to maps of E_n-algebras implies that in the case of $f=id_A$, $HH_{\mathcal{E}_n}(A,A)$ becomes an E_1-algebra object inside the ∞-category E_n-algebra, thus an E_{n+1}-algebra. Our approach uses the relationship between E_n-algebras and factorization algebras which we recall, among other preliminaries, in Section 2. In particular, we recall that E_n-algebras are the same as functors $\mathcal{A}$ from the category of open subsets of $\mathbb{R}^n$ homeomorphic to a disjoint union of disks to chain complexes which are locally constant, *i.e*, such that the natural map $\mathcal{A}(U)\to\mathcal{A}(V)$ is an equivalence when U is a subset of V and both are homeomorphic to a disk. *Factorization homology* on a framed manifold M can be seen as (the left Kan) extension of such a functor to any open set of M (in particular M itself), denoted $(U,\mathcal{A})\mapsto\int_M\mathcal{A}$. In Section 6.1, we recall the relationship between E_n-modules over an E_n-algebra A and factorization homology over $S^{n-1}\times\mathbb{R}$ (endowed with the framing given by its natural embedding in $\mathbb{R}^n$), namely that the category of E_n-A-modules is equivalent to the category of left modules over the (associative) algebra $\int_{S^{n-1}}A$. For $n=\infty$, one recovers that E_∞-A-modules are the same as left modules over A as was proved in [L-HA, Lu2].

Theorem 6.6. *Let A be an E_∞-algebra. There is an equivalence of symmetric monoidal ∞-categories between the category A-Mod^{E_∞} of E_∞ A-Modules and the category of left A-modules (where A is viewed as an E_1-algebra).*

We give a proof of this result using factorization homology in Section 6.2. From this, we deduce in Section 6.3, that, for a closed oriented manifold M, the Poincaré duality isomorphism can be *uniquely lifted* into an E_∞-quasi-isomorphism $C^*(M)\to C_*(M)[\dim(M)]$.

Corollary 6.19. *Let $(X,[X])$ be a Poincaré duality space. The cap-product by $[X]$ induces a quasi-isomorphism of E_∞-$C^*(X)$-modules*

$$C^*(X) \xrightarrow{\simeq} C_*(X)[\dim(X)]$$

realizing the (unique) E_∞-lift of the Poincaré duality isomorphism.

Putting together the above results on Deligne conjecture, Poincaré duality and interpretation of higher Hochschild chains in terms of mapping spaces, we obtain in Section 8 that the chains $C_*(Map(S^n,M))$ for an n-connected manifold M inherits an natural E_{n+1}-algebra structure (Theorem 8.1) which lifts Sullivan-Voronov sphere product in homology. To actually recover the sphere product we use our *explicit* description of the centralizers of E_n-algebra maps. The above results thus yields chain level constructions over any field of coefficient as well as (under the assumptions that M has projective homology groups) over integral coefficient. Actually, results similar to Theorem 8.1 can be obtained using only bimodules maps (and not quasi-isomorphisms) $C^*(M) \to C_*(M)[d]$ (with some properties); this yields a *functorial* construction of E_{n+1}-algebra structures on $C_*(Map(S^n,M))$, see Theorem 8.8.

Furthermore, as another application of Theorems 5.11 and 7.7, in Section 9.1, we apply for an augmented E_∞-algebra A the E_n-algebra structure on higher Hochschild chains $CH^{S^n}(A,k)$ (identified with the centralizer construction for the augmentation $A \to k$) to describe the *iterated Bar construction* of an augmented E_∞-algebra. Indeed, the Bar construction $Bar(A)$ of an E_∞-algebra is naturally an E_∞-algebra which can thus be iterated. With this, we prove that the n-iterated Bar construction $Bar^{(n)}$ is an E_n-coalgebra inside the $((\infty,1))$-category of E_∞-algebras, see Proposition 9.9. We then relate this construction to *iterated loop spaces* by showing that there is an natural map of E_n-coalgebras (and E_∞-algebras) $Bar^{(n)}(C^*(X)) \to C^*(\Omega^n(X))$ which is a quasi-isomorphism if X is n-connected. We also gave similar dual statements for chains on iterated loop spaces using that the dual of the Bar construction is precisely the centralizer $\mathfrak{z}(A \to k) \cong CH^{S^n}(A,k)$ of the augmentation $A \to k$.

Corollary 9.10. *Let Y be a topological space.*

1. *The map $\mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \to C^*\big(\Omega^n(Y)\big)$ is an E_n-coalgebra morphism in the category of E_∞-algebras, which is an equivalence if Y is n-connected.*
2. *Dually, the map $\mathcal{I}t_{\Omega^n} : C_*\big(\Omega^n(Y)\big) \longrightarrow \Big(Bar^{(n)}(C^*(Y))\Big)^\vee$ is an E_n-algebra morphism (in k-Mod_∞). Further, if k is a field, Y is n-connective and has finite dimensional homology groups, then $\Big(Bar^{(n)}(C^*(Y))\Big)^\vee$ is an E_∞-coalgebra and the map $\mathcal{I}t_{\Omega^n}$ is an equivalence of E_n-algebras in E_∞-coAlg.*

In Section 9.2, we consider the *Bar construction of an E_m-algebra.* Using its factorization homology interpretation due to Francis [F1], we prove that it is naturally an E_{m-1}-algebra which allows to iterate this construction up to m-times. Then using the technique of Section 7, we prove that the n-iterated Bar construction of an augmented E_m-algebra ($m \geq 1$) has an natural structure of E_n-coalgebra inside the $((\infty,1)$-)category E_{m-n}-algebras, see Proposition 9.21.

In this paper, we work in Lurie's framework of stable ∞-categories [L-HTT, L-HA], which is very well suited for doing homological algebra in the symmetric monoidal context. In particular we will work over the (derived) $(\infty,1)$-category k-Mod_∞ of chain complexes over a commutative unital ring k. It should be noted that in characteristic zero, one can use CDGA's instead of E_∞-algebras which allows to have (model) categories interpretation of all our results in the spirit of [G, GTZ, GTZ2].

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Conventions and notations:

(1) We use *homological grading*, emphasizing the geometric dimension of the chains on mapping spaces. In particular, unless otherwise stated, differential will lower the degree by one. We will write k-Mod_∞ for the $(\infty,1)$-category of chain complexes of k-modules and $\otimes$ for tensor products over the ground ring k.

(2) We will denote the *Hochschild chain complex* of A, modeled over a space, X with values in an A-module M, by $CH_X(A,M)$ as an object in the stable $(\infty,1)$-category of chain complexes. This is a *covariant* functor in X. Similarly, we will also denote the *Hochschild cochain complex* of A, modeled over a space X, with values in an A-module M, by $CH^X(A,M)$, as an object in the stable $(\infty,1)$-category of chain complexes. This is a *contravariant* functor of X, see § 3.2. This is compatible with the notation introduced in [GTZ2] but not with those in [G, GTZ]. We choose this notation in order to emphasize the variance of the functor with respect to X.

(3) We will denote $HH^{X,n}(A,M)$ and $HH_{X,n}(A,M)$ the degree n homology groups of $CH^X(A,M)$ and $CH_X(A,M)$.

(4) For $n \in \mathbb{N}\cup\{\infty\}$, we will write E_n-*Alg* for the $(\infty,1)$-category of E_n-algebras in k-Mod_∞ as studied in [Lu3, L-HA, F1] We will also denote by $\mathbb{E}_n^{\otimes}$ the ∞-operad governing E_n-algebras, $HH_{\mathcal{E}_n}(A,M)$ for the E_n-Hochschild cohomology of an E_n-algebra with value in an E_n-A-module M (Definition 7.1) and $\int_X A$ for the factorization homology of A on a framed manifold X (see § 2.3). Also $CDGA_\infty$ will be the $(\infty,1)$-category of commutative differential graded k-algebras (CDGA for short).

(5) Given A, an E_n-algebras, we will write A-Mod^{E_n} for the $(\infty,1)$-category of E_n-modules over A. Similarly, if B is an E_m-algebra (with $m\geq n$), we will write B-Mod^{E_n} for the $(\infty,1)$-category of E_n-modules over B viewed as an E_n-algebra.

(6) If A is an E_n-algebra ($n\geq 1$) by a left or right module over A, we mean a left or right module over A viewed as an E_1-algebra. We will denote $A-LMod$ and $A-RMod$ the respective $(\infty,1)$-categories of left and right modules over A.

(7) Unless otherwise stated, we will work in the context of unital algebras.

2. E_n-algebras and factorization homology

In this section we briefly recall notions of $(\infty,1)$-categories, ∞-operads and in particular the E_n-operad and its algebras and their modules. There are several equivalent notions of (symmetric monoidal) $(\infty,1)$-categories and the reader shall feel free to use its favorite ones. Below we recall very briefly one model given by the complete Segal spaces and some examples.

2.1. ∞-categories. Following [R, Lu4], an $(\infty,1)$-category is a *complete Segal space*. There is a simplicial closed model category structure, denoted $\mathcal{S}e\mathcal{S}p$ on the category of simplicial spaces such that a fibrant object in the $\mathcal{S}e\mathcal{S}p$ is precisely a Segal space. Rezk has shown that the category of simplicial spaces has another simplicial closed model structure, denoted $\mathcal{CS}e\mathcal{S}p$, whose fibrant objects are precisely complete Segal spaces [R, Theorem 7.2]. Let $\mathbb{R}: \mathcal{S}e\mathcal{S}p \to \mathcal{S}e\mathcal{S}p$ be a fibrant replacement functor. Let $\widehat{\cdot}: SeSp \to \mathcal{CS}e\mathcal{S}p$, $X_\bullet \to \widehat{X_\bullet}$, be the completion functor that assigns to a Segal space an equivalent complete Segal space. The composition $X_\bullet \mapsto \widehat{\mathbb{R}(X_\bullet)}$ gives a fibrant replacement functor $L_{\mathcal{CS}e\mathcal{S}p}$ from simplicial spaces to complete Segal spaces.

Let us explain how to go from a model category to a simplicial space. The standard key idea is to use Dwyer-Kan localization. Let $\mathcal{M}$ be a model category and $\mathcal{W}$ be its subcategory of weak-equivalences. We denote $L^H(\mathcal{M},\mathcal{W})$ its *hammock localization*, see [DK]. One of the main property of $L^H(\mathcal{M},\mathcal{W})$ is that it is a simplicial category and that the (usual) category $\pi_0(L^H(\mathcal{M},\mathcal{W}))$ is the homotopy category of $\mathcal{M}$. Further, every weak equivalence has a (weak) inverse in $L^H(\mathcal{M},\mathcal{W})$. When $\mathcal{M}$ is further a simplicial model category, then for every pair (x,y) of objects $Hom_{L^H(\mathcal{M},\mathcal{W})}(x,y)$ is naturally homotopy equivalent to the derived mapping space $\mathbb{R}Hom(x,y)$.

It follows that any model category $\mathcal{M}$ gives functorially rise to the simplicial category $L^H(\mathcal{M},\mathcal{W})$. Taking the nerve $N_\bullet(L^H(\mathcal{M},\mathcal{W}))$ we obtain a simplicial space. Composing with the complete Segal Space replacement functor we get a functor $\mathcal{M} \to L_\infty(\mathcal{M}) := L_{\mathcal{CS}e\mathcal{S}p}(N_\bullet(L^H(\mathcal{M},\mathcal{W})))$ from model categories to $(\infty,1)$-categories (that is complete Segal spaces).

Example 2.1. Applying the above procedure to the model category of simplicial sets $sSet$, we obtain the $(\infty,1)$-category $sSet_\infty$. Similarly from the model category $CDGA$ of commutative differential graded algebras, which we refered to as CDGAs for short, we obtain the $(\infty,1)$-category $CDGA_\infty$. Note that a simplicial space is determined by its $(\infty,0)$ path groupoid and therefore the category of simplicial sets should be thought of as the $(\infty,1)$ category of all $(\infty,0)$ groupoids. Further, the disjoint union of simplicial sets and the tensor products (over k) of algebras are monoidal functors which gives $sSet$ and $CDGA$ a structure of monoidal model category (see [Ho] for example). Thus $sSet_\infty$ and $CDGA_\infty$ also inherit the structure of symmetric monoidal $(\infty,1)$-categories in the sense of [R, Lu4].

The model category of topological spaces yields the $(\infty,1)$-category Top_∞. Since $sSet$ and Top are Quillen equivalent [GJ, Ho], their associated $(\infty,1)$-categories are equivalent (as $(\infty,1)$-categories): $sSet_\infty \underset{\sim}{\overset{\sim}{\rightleftarrows}} Top_\infty$, where the left and right equivalences are respectively induced by the singular set and geometric realization functors.

One can also consider the pointed versions $sSet_{\infty *}$ and $Top_{\infty *}$ of the above $(\infty, 1)$-categories (using the model categories of these pointed versions [Ho]).

Example 2.2. There are model categories categories A-Mod and A-$CDGA$ of modules and commutative algebras over a CDGA A, thus the above procedure gives us $(\infty, 1)$-categories A-Mod_∞ and A-$CDGA_\infty$ and the base changed functor lifts to an $(\infty, 1)$-functor. Further, if $f : A \to B$ is a weak equivalence, the natural functor $f_* : B\text{-}Mod \to A\text{-}Mod$ induces an equivalence $B\text{-}Mod_\infty \overset{\sim}{\to} A\text{-}Mod_\infty$ of $(\infty, 1)$-categories since it is a Quillen equivalence.

Moreover, if $f : A \to B$ is a morphism of CDGAs, it induces a natural functor $f^* : A\text{-}Mod \to B\text{-}Mod, M \mapsto M \otimes_A B$, which is an equivalence of $(\infty, 1)$-categories when f is a quasi-isomorphism, and is a (weak) inverse of f_* (see [TV] or [KM]). Here we also (abusively) denote $f^* : A\text{-}Mod_\infty \to B\text{-}Mod_\infty$ and $f_* : B\text{-}Mod_\infty \to A\text{-}Mod_\infty$ the (derived) functors of $(\infty, 1)$-categories induced by f. Since we are working over a field of characteristic zero, the same results applies to monoids in A-Mod and B-Mod, that is to the categories $A - CDGA_\infty$ and $B - CDGA_\infty$.

Also, note that if A, B, C are CDGAs and $f : A \to B$, $g : A \to C$ are CDGAs morphisms, we can form the (homotopy) pushout $D \cong B \otimes^{\mathbb{L}}_A C$; let us denote $p : B \to D$ and $q : C \to D$ the natural CDGAs maps. Then we get the two natural based change $(\infty, 1)$-functors $C\text{-}Mod_\infty \underset{p_* \circ q^*}{\overset{f^* \circ g_*}{\rightrightarrows}} B\text{-}Mod_\infty$. Given any $M \in C\text{-}Mod$, the natural map $f^* \circ g_*(M) \to p_* \circ q^*(M)$ is an equivalence [TV, Proposition 1.1.0.8].

2.2. **∞-operads, E_n-algebras and their modules.** An operad is a special case of a colored operad which itself is a special case of an ∞-operad. An *infinity operad* $\mathcal{O}$ is an ∞-category together with a functor $\mathcal{O}^\otimes \to N(Fin_*)$ satisfying a list of axioms, see [L-HA]. The simplest example of an ∞-operad is $N(Fin_*) \to N(Fin_*)$. This example is the ∞-operad associated to the operad $Comm$. In other words $Comm^\otimes = N(Fin_*)$. The configuration spaces of small n-dimensional cubes embedded in a bigger n-cube form an operad, $\mathbb{E}_n$, whose associated ∞-operad is denoted by $\mathbb{E}_n^\otimes$ see [L-HA]. This example as the same objects as Fin_*, and we will denote $\mathbb{E}_n^\otimes(I, J)$ its spaces of morphisms from I to J. There is a standard model for this operad given by $\big(\mathcal{C}_n(r)\big)_{r\geq 1}$, the *operad of dimension n-cubes*, see [Ma, L-HA], where $\mathcal{C}_n(r)$ is the configuration space of rectilinear embeddings of r-disjoint cubes inside an unit cube. Its singular chain $C_*(\mathcal{C}_n(r))$ is a model for the operad governing E_n-algebras in k-Mod$_\infty$.

Recall that, for any integer $n \geq 0$, $\mathbb{E}_n^\otimes$ denotes the ∞-operad of little n-cubes. By an $\mathbb{E}_n$-algebra we mean an algebra over the ∞-operad $\mathbb{E}_n^\otimes$. Let E_n-Alg denotes the symmetric monoidal ∞-category of $\mathbb{E}_k$-algebras in the symmetric monoidal ∞-category of differential graded k-modules. For any $\mathbb{E}_n$-algebra A, let $A\text{-}Mod^{E_n}$ denote the symmetric monoidal ∞-category of modules over an $\mathbb{E}_n$-algebra A. If $\mathcal{C}$ is a symmetric monoidal $(\infty, 1)$-category (different from k-Mod$_\infty$), we denote $E_n\text{-}Alg(\mathcal{C})$ for the $(\infty, 1)$-category of *E_n-algebras in $\mathcal{C}$* and similarly $E_n\text{-}coAlg(\mathcal{C})$ for the category of *E_n-coalgebras in $\mathcal{C}$*.

There are natural maps (sometimes called the stabilization functors)

$$\mathbb{E}_0^\otimes \longrightarrow \mathbb{E}_1^\otimes \longrightarrow \mathbb{E}_2^\otimes \longrightarrow \cdots \tag{2}$$

(induced by taking products of cubes with the interval $(-1, 1)$). It is a fact that the colimit of this diagram, denoted by $\mathbb{E}_\infty^\otimes$ can be identified with $Comm^\otimes$ [L-HA,

Section 5.1]. In particular, for any $n\in\mathbb{N}-\{0\}\cup\{+\infty\}$, the map $\mathbb{E}_1^{\otimes}\to\mathbb{E}_n^{\otimes}$ induces a functor $E_n\text{-}Alg\to E_1\text{-}Alg$ which associates to an E_n-algebra its underlying E_1-algebra structure.

According to Lurie [L-HA] (also see [F1, AFT]), we also have an alternative definition for E_n-algebras.

Definition 2.3. The $(\infty,1)$-category of *E_n-algebras*, is defined as the ($(\infty,1)$-)category of symmetric monoidal functors

$$Fun^{\otimes}(Disk_n^{fr},\, k\text{-}\mathrm{Mod}_\infty)$$

where $Disk_n^{fr}$ is the category with objects the integers and morphism the spaces $Disk_n^{fr}(k,\ell):=Emb^{fr}(\coprod_k\mathbb{R}^n,\coprod_\ell\mathbb{R}^n)$ of framed embeddings of k disjoint copies of a disk $\mathbb{R}^n$ into ℓ such copies; the monoidal structure is induced by disjoint union of copies of disks.

We will denote by $Map_{E_n\text{-}Alg}(A,B)$ the mapping space of E_n-algebras maps from A to B, *i.e.*, the space of maps between the associated symmetric monoidal functors.

In other words, $E_n\text{-}Alg$ is equivalent to the $(\infty,1)$-category of $Disk_n^{fr}$-algebras (where $Disk_n^{fr}$ is equipped with its obvious ∞-operad structure). The tensor products in $k\text{-}\mathrm{Mod}_\infty$ induces a symmetric monoidal structure on $E_n\text{-}Alg$ as well (which, for instance, can be represented by usual Hopf operads such as those arising from the filtration of the Barratt-Eccles operad [BF]).

Example 2.4 (Opposite of an E_n-algebra)**.** There is a canonical $\mathbb{Z}/2\mathbb{Z}$-action on $E_n\text{-}Alg$ induced by the antipodal map $\tau:\mathbb{R}^n\to\mathbb{R}^n$, $x\mapsto -x$ acting on the source of $Fun^{\otimes}(Disk_n^{fr},\, k\text{-}\mathrm{Mod}_\infty)$. If A is an E_n-algebra, then the result of this action $A^{op}:=\tau^*(A)$ is its opposite algebra. If $n=\infty$, the antipodal map is homotopical to the identity so that A^{op} is equivalent to A as an E_∞-algebra.

Example 2.5. Let X be a topological space. Then its *singular cochain complex $C^*(X)$ has an natural structure of E_∞-algebras*, whose underlying E_1-structure is given by the usual (strictly associative) cup-product (for instance see [M2]). Similarly, the *singular chains $C_*(X)$ have an natural structure of E_∞-coalgebra* which is the predual of $(C^*(X),\cup)$. There are similar explicit constructions for simplicial sets $X_\bullet$ instead of spaces, see [BF]. We recall that $C^*(X)$ is the linear dual of the singular chain complex $C_*(X)$ which is the geometric realization (in $k\text{-}\mathrm{Mod}_\infty$) of the simplicial k-module $k[\Delta_\bullet(X)]$ spanned by the singular set $\Delta_\bullet(X):=Map(\Delta^\bullet,X)$, where Δ^n is the standard n-dimensional simplex. Note that, for E_∞-algebras A, B, the mapping space $Map_{E_\infty\text{-}Alg}(A,B)$ is the (geometric realization of the) simplicial set $[n]\mapsto Hom_{E_\infty\text{-}Alg}\big(A,B\otimes C^*(\Delta^n)\big)$.

Remark 2.6. The $(\infty,1)$-category $E_\infty\text{-}Alg$ is enriched over $sSet_\infty\cong Top_\infty$ and has all (∞-)colimits. In particular, it is *tensored* over $sSet_\infty$, see [L-HTT, L-HA] for details on tensored ∞-categories (and [Ke] for the classical theory) or, for instance, [EKMM, MCSV] in the E_∞-case (in the context of topologically enriched model categories). We recall that it means that there is a functor $E_\infty\text{-}Alg\times sSet_\infty\to E_\infty\text{-}Alg$, denoted $(A,X_\bullet)\mapsto A\boxtimes X_\bullet$, together with natural equivalences

$$Map_{E_\infty\text{-}Alg}\big(A\boxtimes X_\bullet,B\big)\;\cong\;Map_{sSet_\infty}\big(X_\bullet,Map_{E_\infty\text{-}Alg}(A,B)\big).$$

Note that the tensor $A \boxtimes X_\bullet$ can be computed as the colimit $\varinjlim p^A_{X_\bullet}$ where $p^A_{X_\bullet}$ is the constant map $X_\bullet \to E_\infty\text{-}Alg$ taking value A, for instance see [L-HTT, Corollary 4.4.4.9]. Similarly, $CDGA_\infty$ is tensored over $sSet_\infty$ (and thus Top_∞ as well).

We will use the following fact, which identifies the coproduct in E_∞-Alg with the tensor product, to show the Hochschild complex of an E_∞-algebra model over a space X has a natural E_∞ structure.

Proposition 2.7. *In the symmetric monoidal* $(\infty, 1)$*-category* E_∞*-Alg, the tensor product is a coproduct.*

For a proof see Proposition 3.2.4.7 of [L-HA] (or [KM, Corollary 3.4]); this essentially follows from the observation that an E_∞-algebra is a commutative monoid in (k-Mod$_\infty, \otimes$), see [L-HA] or [KM, Section 5.3]. In particular, Proposition 2.7 implies that, for any finite set I, $A^{\otimes I}$ has an natural structure of E_∞-algebras which can be rephrased as

Proposition 2.8. *A symmetric monoidal functor* $N(Fin) \to k\text{-}Mod_\infty$ *has a natural lift to a functor* $N(Fin) \to E_\infty\text{-}Alg$*.*

It follows that, for a finite set I, we have an natural multiplication map

$$A^{\otimes I} \stackrel{m_A^{(I)}}{\longrightarrow} A$$

which is a map in E_∞-Alg and is compatible with compositions. We will simply write $m_A : A \otimes A \to A$ for the E_∞-algebra map obtained by taking $I = \{0, 1\}$.

2.3. **Factorization algebras and Factorization homology.** Given a topological manifold M of dimension n one can define a colored operad whose objects are open subsets of M that are homeomorphic to $\mathbb{R}^n$ and whose morphisms from $\{U_1, \cdots, U_n\}$ to V are empty unless when U_i's are mutually disjoint subsets of U, in which case they are singletons. The ∞-operad associated to this colored operad is denoted by $N(\mathrm{Disk}(M))$, see [L-HA], Remark 5.2.4.7. Unfolding the definition we find that an $N(\mathrm{Disk}(M))$-algebra on a manifold M, with value in chain complexes, is a rule that assigns to any open disk[2] U a chain complex $\mathcal{F}(U)$ and, to any finite family of disjoint open disks $U_1, \ldots, U_n \subset V$ included in a disk V, a natural map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$. An $N(\mathrm{Disk}(M))$-algebra is **locally constant** if for any inclusion of open disks $U \hookrightarrow V$ in X, the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism (see [L-HA, Lu3]) .

Locally constant $N(\mathrm{Disk}(M))$-algebra are actually (homotopy) *locally constant factorization algebras* in the sense of Costello [CG, C], see Remark 2.13 below. We thus denote Fac^{lc}_M *the* $(\infty, 1)$*-category of locally constant* $N(Disk(M))$*-algebras.* We will also denote $N(\mathrm{Disk}(M)$-Alg the $(\infty, 1)$-category of $N(\mathrm{Disk}(M)$-algebras.

If $\mathcal{A}$ is a locally constant $N(\mathrm{Disk}(M))$-algebra, the rule which to an open disk D associates the chain complex $\mathcal{A}(D)$ can be extended to any open set of M. In fact, Lurie has proved [L-HA, Lu3] that the functor $\mathrm{Disk}(M) \xrightarrow{\mathcal{A}} k\text{-Mod}_\infty$ has a left Kan extension along the embedding $\mathrm{Disk}(M) \hookrightarrow Op(M)$ where $Op(M)$ is the standard ($(\infty, 1)$-)category of open subsets of M, *i.e.*, with objects the open subsets of M and morphism from U to V are empty unless when $U \subset V$ in which case they are singletons.

[2] i.e. an open subset of M homeomorphic to a euclidean ball

Definition 2.9. Let M be a topological manifold and $\mathcal{A}$ be a locally constant factorization algebra.

Factorization homology is the $(\infty,1)$-functor $Op(M)\otimes \mathrm{Fac}_M^{lc} \to k\text{-Mod}_\infty$, denoted $(M,\mathcal{A})\mapsto \int_M \mathcal{A}$, given by the left Kan extension of $\mathrm{Disk}(M) \xrightarrow{\mathcal{A}} k\text{-Mod}_\infty$. We say that $\int_M \mathcal{A}$ is the factorization homology of M with values in $\mathcal{A}$.

Remark 2.10. Factorization homology is also called topological chiral homology in [L-HA, Lu4, Lu3]. We prefer Francis terminology [F1, AFT, F2] which is justified by the fact that factorization homology is actually the homology (or said otherwise derived sections) of factorizations algebras in the sense of Costello [CG] as we proved [GTZ2], see Remark 2.13 below.

Example 2.11. If M is a framed manifold, then any E_n-algebra is a locally constant factorization algebra on M (for instance, see [L-HA, GTZ2]).

Example 2.12. The canonical map $N(\mathrm{Disk}(M))\to N(Fin_*)$ shows that any E_∞-algebras has a canonical structure of locally factorization algebra on any topological manifold M. This structure is studied in details (using the Hochschild chain models) in [GTZ2] and actually extends to define factorization algebra on any CW-complex.

Remark 2.13. Let us justify a bit more the terminology of locally constant factorization algebras we are using (hoping it will avoid any possible confusion). The notion of locally constant $N(\mathrm{Disk}(X))$-algebra is actually equivalent to the "full" notion of a locally constant factorization algebra on X in the sense of Costello [CG, C] which are a a similar construction where the U_i are allowed to be any open subsets, satisfying a kind of "Čech/cosheaf-like" condition (and still being locally constant in the above sense). Let us now be more precise. A **prefactorization** algebra is an algebra over the colored operad whose objects are open subsets of X and whose morphisms from $\{U_1,\cdots,U_n\}$ to V are empty unless when U_i's are mutually disjoint subsets of U, in which case they are singletons. Unfolding the definition we find that a prefactorization algebra on X, with value in chain complexes, is a rule that assigns to any open set U a chain complex $\mathcal{F}(U)$ and, to any finite family of pairwise disjoint open sets $U_1,\dots,U_n\subset V$ included in an open V, a natural map $\mathcal{F}(U_1)\otimes\cdots\otimes\mathcal{F}(U_n)\to\mathcal{F}(V)$. These structures maps are required to satisfy obvious associativity and symmetry conditions, see [CG]. These structure maps allows to define "Čech-complexes" associated to a cover $\mathcal{U}$ of U. Denoting $P\mathcal{U}$ the set of finite pairwise disjoint open subsets $\{U_1,\dots,U_n\,,\,U_i\in\mathcal{U}\}$, it is, by definition the chain (bi-)complex

$$\check{C}(\mathcal{U},\mathcal{F}) = \bigoplus_{P\mathcal{U}} \mathcal{F}(U_1)\otimes\cdots\otimes\mathcal{F}(U_n) \leftarrow \bigoplus_{P\mathcal{U}\times P\mathcal{U}} \mathcal{F}(U_1\cap V_1)\otimes\cdots\otimes\mathcal{F}(U_n\cap V_m)\leftarrow\cdots$$

where the horizontal arrows are induced by the alternate sum of the natural inclusions as for the usual Čech complex of a cosheaf (see [CG]). The prefactorization algebra structure also induce a canonical map $\check{C}(\mathcal{U},\mathcal{F})\to\mathcal{F}(U)$.

A prefactorization algebra $\mathcal{F}$ on X is said to be a **factorization algebra** (in the sense of [CG]) if, for all open subset $U\in Op(X)$ and every factorizing cover[3] $\mathcal{U}$ of U, the canonical map $\check{C}(\mathcal{U},\mathcal{F})\to\mathcal{F}(U)$ is a quasi-isomorphism (see [C, CG]).

[3]an open cover of $\mathcal{U}$ is factorizing if, for all finite collections $x_1,\dots,x_n$ of distinct points in U, there are pairwise disjoint open subsets $U_1,\dots,U_k$ in $\mathcal{U}$ such that $\{x_1,\dots,x_n\}\subset\bigcup_{i=1}^k U_i$

Again, a factorization algebra is *locally constant* if for any inclusion of open disks $U \hookrightarrow V$ in X, the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism.

In [GTZ2], we proved

Theorem 2.14 ([GTZ2, Theorem 6]). *The functor* $(U, \mathcal{A}) \mapsto \int_U \mathcal{A}$ *induces an equivalence of* $(\infty, 1)$*-categories between locally constant* $N(\mathrm{Disk}(X))$*-algebra and locally constant factorization algebras on the manifold* X *in the sense of* [CG]. *Further this functor is (equivalent to) the identity functor when restricted to open disks.*

This justifies our terminology of locally constant factorization algebras and factorization homology; further, the extension on any open set U of a (locally constant) $N(\mathrm{Disk}(X))$-algebra $\mathcal{A}$ is precisely given by the factorization homology $\int_U \mathcal{A}$, see *loc. cit.*.

If U is an open subset of X, then a (locally constant) factorization algebra $\mathcal{A}$ on X has a canonical restriction $\mathcal{A}_{|U}$ into a (locally constant) factorization algebra on U. Furthermore, if $\mathcal{U} = (U_i)_{i \in I}$ is a cover of X, the factorization algebra $\mathcal{A}$ on X can be uniquely recovered by the data of the factorization algebras $\mathcal{A}_{|U_i}$ restricted to the U_i's (thanks to the Čech condition applied to suitable covers). In fact, any family of factorization algebras $\mathcal{F}_i$ on U_i, satisfying natural compatibility conditions on the intersections of the U_i's, extends uniquely into a factorization algebra on X; we refer to Costello-Gwilliam [CG, Section 4] for details on this descent property of factorization algebras.

2.4. **E_n-algebras as factorization algebras.** Theorem 5.2.4.9 of [L-HA] (also see [Lu3, GTZ2]) provides an equivalence between E_n-algebras and *locally constant factorization algebra* on the open disk D^n:

Proposition 2.15. *There is a natural equivalence of* $(\infty, 1)$*-categories*

$$E_n\text{-}Alg \;\cong\; Fac^{lc}_{\mathbb{R}^n}.$$

In particular, an E_n-algebra can be seen as an n-dimensional (topological) field theory (over the space-time manifold $\mathbb{R}^n$), providing invariant for framed n-manifolds precisely computed by Factorization homology.

Let $\pi : X \times Y \to X$ be the canonical projection. There is the *pushforward* functor $\pi_* : \mathrm{Fac}^{lc}_{X \times Y} \to \mathrm{Fac}^{lc}_X$ defined, for U open (disk) in X and $\mathcal{A} \in \mathrm{Fac}^{lc}_{X \times Y}$, by $\pi_*(\mathcal{A})(U) := \mathcal{A}(\pi^{-1}(U)) = \mathcal{A}(U \times Y)$; see [CG] for details on pushforward for factorization algebras. Here the fact that the pushforward of a locally constant factorization algebra is locally constant follows from the fact that the fibers are all naturally identified with the same manifold Y. The fact that locally constant factorization algebras on $\mathbb{R}^n$ are E_n-algebras implies that, when $Y = \mathbb{R}^n$, the pushforward factors through a functor $\pi_* : \mathrm{Fac}^{lc}_{X \times \mathbb{R}^n} \to \mathrm{Fac}^{lc}_X(E_n\text{-}Alg)$ see [GTZ2]. In particular, we can take $X = \mathbb{R}^m$. The following ∞-category version of Dunn Theorem was proved by Lurie [L-HA] (and [GTZ2] for the pushforward interpretation):

Theorem 2.16 (Dunn Theorem). *There is an equivalence of* $(\infty, 1)$*-categories* $E_{m+n}Alg \cong E_n\text{-}Alg(E_m\text{-}Alg)$. *Under the equivalence* $E_n\text{-}Alg \cong Fac^{lc}_{\mathbb{R}^n}$ *(Proposition 2.15), the above equivalence is realized by the pushforward* $\pi_* : Fac^{lc}_{\mathbb{R}^m \times \mathbb{R}^n} \to Fac^{lc}_{\mathbb{R}^m}(E_n\text{-}Alg)$ *associated to the canonical projection* $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$.

Let Fin and Fin_* denote the categories of finite sets and pointed finite sets respectively. There is a forgetful functor $Fin_* \to Fin$ forgetting which point is the base point. There is also a functor $Fin \to Fin_*$ which adds an extra point called the base point.

Further, since the ∞-operads $\mathbb{E}_n^{\otimes}$ are coherent (see [L-HA, Lu3]), the *categories* A-Mod^{E_n} *for* $A \in E_n$-Alg *assembles to form an* $(\infty,1)$-*category* of all E_n-algebras and their modules, denoted Mod^{E_n} (or $Mod^{E_n}(\mathcal{C})$ when we want to emphasize $\mathcal{C}$). The canonical functor $Fin \to Fin_*$ adding a base point yields a canonical functor [4] $\iota : Mod^{E_n}(\mathcal{C}) \to Alg_{E_n}(\mathcal{C})$ which gives rise, for any E_n-algebra A, to a (homotopy) pullback square:

$$\begin{tikzcd} A\text{-}Mod^{E_n} \arrow[r] \arrow[d] & Mod^{E_n} \arrow[d, "\iota"] \\ \{A\} \arrow[r] & E_n\text{-}Alg \end{tikzcd} \tag{3}$$

We refer to [Lu2, L-HA, F1] for details. Note that the functor ι is monoidal. Further, if $A \xrightarrow{f} B$, $A \xrightarrow{g} C$ are two maps of E_∞-algebras, and $M \in B$-Mod^{E_∞} and $N \in C$-Mod^{E_∞}, then

$$\iota\Big(M \overset{\mathbb{L}}{\underset{A}{\otimes}} N\Big) \quad \cong \quad B \overset{\mathbb{L}}{\underset{A}{\otimes}} C. \tag{4}$$

Example 2.17. if $n = 1$, A-Mod^{E_1} is equivalent to the $(\infty,1)$-category of A-bimodules and if $n = \infty$, A-Mod^{E_∞} is equivalent to the $(\infty,1)$-category of left A-modules, see [Lu2, L-HA] (and Proposition 6.1, Theorem 6.6 below). In general, A-Mod^{E_n} can be described in terms of factorization homology of A, see § 6.1.

One can define a notion of locally constant factorization algebra for stratified manifolds and factorization homology as well. We refer to [AFT] for details. In this paper, we will only need a very special and easy case; the disk with a special point. Let D^n_* be the pointed Euclidean open disk viewed as a stratified manifold with only two strata: a dimension 0 stratum given by its center and the dimension n-stratum given by the disk.

Definition 2.18. We say that a factorization algebra (or an $N(\mathrm{Disk}(D^n)$-algebra) on D^n_* is locally constant if the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is an equivalence when $U \subset V$ are open disks such that either U contains the base point or V is included in the n-stratum $D^n - \{0\}$. In other words, we do *not* require $\mathcal{F}(U) \to \mathcal{F}(V)$ to be an equivalence if V contains the base point while U does not. We denote $Fac^{lc}_{D^n_*}$, the $(\infty,1)$-category consisting of a pair $(\mathcal{M},\mathcal{A})$ of locally constant factorization algebras on respectively D^n_* and D^n together with an isomorphism of factorization algebras $\mathcal{M}_{D^n-\{0\}} \overset{\simeq}{\to} \mathcal{A}_{D^n-\{0\}}$ between the restrictions of $\mathcal{M}$ and $\mathcal{A}$ to $D^n - \{0\}$.

Remark 2.19. The factorization algebra $\mathcal{A}$ is actually determined by $\mathcal{M}$, since, by the locally constant condition, it is essentially defined by its restriction to any open ball in D^n, thus to any open ball included in $D^n - \{0\}$. In particular, for $n > 1$, the category of locally constant factorization algebras on the stratified disk D^n_* is equivalent to $\mathrm{Fac}^{lc}_{D^n_*}$, which justifies our notation. For $n = 1$, the latter category is equivalent to the category of all bimodules (over an E_1-algebras), while the first

[4] which essentially forget the underlying module

one is equivalent to the category of (A,B)-bimodules where A,B may be different E_1-algebras.

Proposition 2.15 extends immediately to

Proposition 2.20. *There is an equivalence between the $(\infty,1)$-categories Mod^{E_n} of all E_n-modules and $Fac^{lc}_{D^n_*}$, the locally constant factorization algebras on the pointed disk as in Definition 2.18.*

Note that the pushforward $D^n_* \to *$ realizes the forgetful functor $Mod^{E_n}(\mathcal{C}) \to \mathcal{C}$ and further, as noted in Remark 2.19, fixing an euclidean sub-disk $D \subset D^n_* - \{0\}$ we get a functor $Fac^{lc}_{D^n_*} \to Fac^{lc}_{D}$ which is equivalent to the functor $\iota : Mod^{E_n} \to E_n\text{-}Alg$, *i.e.*, to the forgetful functor $(\mathcal{M},\mathcal{A}) \mapsto \mathcal{A}$. Forgetting the stratification yield a functor $Fac^{lc}_{D^n} \to Fac^{lc}_{D^n_*}$ realizing the canonical functor $E_n\text{-}Alg \to Mod^{E_n}$ (which view an E_n-algebra as a module over itself in a canonical way).

Remark 2.21. Let A be an E_n-algebra and $f : A \to B$ an E_n-algebra map and let B be endowed with the induced A-E_n-module structure. This module structure has an easy description in terms of factorization algebras. Denote $\mathcal{A} : U \mapsto \int_U A$ and $\mathcal{B} : V \mapsto \int_V B$ be the associated factorization algebras on $\mathbb{R}^n$ (see Theorem 2.14). By Proposition 2.20 and § 2.2, the data of the A-E_n-module structure on B is equivalent to the data which, to any embedding $\coprod_{i=0}^r \phi_i : \coprod_{i=0}^r \mathbb{R}^n \to \mathbb{R}^n$ (with $\phi_0(0)=0$) and commutative diagram

$$\begin{tikzcd}
\coprod_{i=0}^r \mathbb{R}^n \arrow[rr, "h"] \arrow[dr, "\coprod_{i=0}^r \phi_i"'] & & \mathbb{R}^n \arrow[dl, "\psi"] \\
& \mathbb{R}^n &
\end{tikzcd}$$

of embeddings, associate an natural[5] map

$$\mathcal{A}(\phi_1(\mathbb{R}^n)) \otimes \cdots \otimes \mathcal{A}(\phi_r(\mathbb{R}^n)) \otimes \mathcal{B}(\phi_0(\mathbb{R}^n)) \longrightarrow \mathcal{B}(\psi(\mathbb{R}^n)). \tag{5}$$

This map (5) is very simple to describe, it is the composition

$$\int_{\phi_1(\mathbb{R}^n)} A \otimes \cdots \otimes \int_{\phi_r(\mathbb{R}^n)} A \otimes \int_{\phi_0(\mathbb{R}^n)} B$$
$$\xrightarrow{\left(\bigotimes_{i=1}^r \int_{\phi_i(\mathbb{R}^n)} f\right)\otimes id} \int_{\phi_1(\mathbb{R}^n)} B \otimes \cdots \otimes \int_{\phi_r(\mathbb{R}^n)} B \otimes \int_{\phi_0(\mathbb{R}^n)} B$$
$$\longrightarrow \int_{\psi(\mathbb{R}^n)} B \cong \mathcal{B}(\psi(\mathbb{R}^n)).$$

where the last map is given by the factorization algebra structure of $\mathcal{B}$, *i.e.*, the E_n-algebra structure of B. Now, let $g : B \to C$ be another E_n-algebra map endowing C with an A-E_n-module structure; let $\mathcal{C} : U \mapsto \int_U C$ be the associated factorization algebra. Then a map A-E_n-modules $h : B \to C$ is equivalent to the data of a stratified factorization algebra map $\int_U h : \mathcal{B}(U) \cong \int_U B \to \int_U C \cong \mathcal{C}(U)$ such that,

[5]with respect to composition of embeddings, that is satisfies the usual associativity condition of the structure maps of a prefactorization algebra in the sense of [CG]

for all $\phi_0, \dots, \phi_r$ and ψ as above, the following diagram

$$\begin{tikzcd}
\Big(\bigotimes_{i=1}^r \int_{\phi_i(\mathbb{R}^n)} A\Big) \otimes \int_{\phi_0(\mathbb{R}^n)} B \arrow[r, "(\bigotimes \int_{\phi_i(\mathbb{R}^n)} f)\otimes id"] \arrow[d, "(\bigotimes id)\otimes \int_{\phi_0(R^n)} h"'] & \bigotimes_{i=0}^r \int_{\phi_i(\mathbb{R}^n)} B \arrow[r] & \int_{\psi(\mathbb{R}^n)} B \arrow[d, "\int_{\psi(\mathbb{R}^n)} h"] \\
\Big(\bigotimes_{i=1}^r \int_{\phi_i(\mathbb{R}^n)} A\Big) \otimes \int_{\phi_0(\mathbb{R}^n)} C \arrow[r, "(\bigotimes \int_{\phi_i(\mathbb{R}^n)} g\circ f)\otimes id"'] & \bigotimes_{i=0}^r \int_{\phi_i(\mathbb{R}^n)} C \arrow[r] & \int_{\psi(\mathbb{R}^n)} C
\end{tikzcd}$$

is commutative.

3. Higher Hochschild (co)chains for E_∞-algebras

In this section we define and study higher Hochschild (co)chains modeled over spaces for E_∞-algebras with values in E_∞-modules.

3.1. Factorization homology of E_∞-algebras and higher Hochschild chains.

Factorization homology with values in E_∞-algebras has special properties. It becomes an homotopy invariant and can be defined over any space, providing an homology theory for spaces. Indeed, it identifies with the Hochschild chains modeled on a space and with values in an E_∞-algebra, see Theorem 3.11 below. We will denote $(X, A) \mapsto CH_X(A)$ the latter construction which we explain further in this section. The reader familiar with factorization homology for commutative algebras can skip it and keep in mind that $CH_X(A)$ means factorization homology extended to spaces.

The Hochschild chains $CH_{X_\bullet,\bullet}(A, A)$ over a simplicial set $X_\bullet$ with value in a CDGA A is defined in [P]. As explained in [GTZ2], it can be defined using the PROPic definition of commutative (differential graded) k-algebras as follows. A CDGA over k is a strict symmetric monoidal functor $A : Fin \to k$-Mod from the category of finite sets (with disjoint union for the monoidal structure) to the category of chain complexes (with tensor product as the monoidal structure). Given a finite simplicial set $\Delta^{op} \to Fin$ we can compose these two functors to get a simplicial k-module. The geometric realization of this simplicial k-module is the Hochschild complex modeled on X. In fact one can do better. A strictly symmetric monoidal functor $A : Fin \to k$-Mod has a canonical lift to $A : Fin \to k$-Alg, along the forgetful functor k-Alg $\to k$-Mod. This is due the simple fact that for a commutative algebra A, the multiplication $m : A \otimes A \to A$ is a map of algebras or said otherwise the fact that the tensor product is a coproduct in CDGA. Thus, the above procedure gives rise to a simplicial CDGA whose geometric realization is the Hochschild complex modeled on $X_\bullet$, with a canonical CDGA structure. In the classical example $X = S^1$, whereby we get the classical Hochschild complex, this is the shuffle product on $CH_\bullet(A, A)$. Another way of saying this is to say that that $CDGA$ is tensored over simplicial sets and that Hochschild chains over $X_\bullet$ with values in A is simply the tensor $A \boxtimes X_\bullet$ (see [L-HA, L-HTT, EKMM, Ke, MCSV] for details on tensored categories over spaces and Remark 2.6). We now discuss how all of the above generalize to the case of E_∞-algebras.

Recall from Section 2 that the $(\infty, 1)$-category of E_∞-algebras (with value in chain complexes) is equivalent to the $(\infty, 1)$-category $Fun^\otimes(N(Fin), k\text{-Mod}_\infty)$ of

(lax) monoidal functors from (the ∞-category associated to) Fin to the $(\infty,1)$-category of chain complexes (see Lurie [Lu2, L-HA, KM]). We denote

$$\Big(X \mapsto CH_X^{simp}(A)\Big) \in Fun^{\otimes}(N(Fin), k\text{-Mod}_\infty)$$

the[6] monoidal functor associated to an E_∞-algebra A. This functor extends naturally to a functor $Fun^{\otimes}(N(Set), k\text{-Mod}_\infty)$ (where Set is the category of sets) by taking colimits that is to say $X \mapsto \varinjlim_{Fin\ni K\to X} CH_K^{simp}(A)$. By Proposition 2.8, $CH_K^{simp}(A)$ has an natural structure of E_∞-algebras.

Remark 3.1. Fixing a set X, $CH_X(A)$ is (functorially) quasi-isomorphic to the tensor product $A^{\otimes X}$ (where A is viewed as a chain complex). Note that this construction (of the underlying chain complex structure) is the same as the one in[GTZ, Section 2.1] in the case of CDGAs. However the functorial structure involves higher homotopies and not only the multiplication and seems difficult to write explicitly on this particular choice of cochain complex.

Let $DK : sk\text{-}Mod_\infty \to k\text{-Mod}_\infty$ be the Dold-Kan functor from the $(\infty,1)$-category of simplicial k-modules to the chain complexes. The Dold-Kan functor refines to a functor $sE_\infty\text{-}Alg \to E_\infty\text{-}Alg$ from simplicial E_∞-algebras to differential graded E_∞-algebras which preserves weak-equivalences (see [M1, Section 3]).

Definition 3.2. The derived Hochschild chain of an E_∞-algebra A and a simplicial set $X_\bullet$ is

$$CH_{X_\bullet}(A) := DK\Big(\varinjlim_{Fin\ni K\to X_\bullet} CH_K^{simp}(A)\Big).$$

Remark 3.3. In the case where the E_∞-algebra A is strict, *i.e.* a CDGA, it follows from Corollary 3.7 below that $CH_{X_\bullet}(A)$ is quasi-isomorphic to the Hochschild chain complex over $X_\bullet$ described in details in[GTZ, Section 2.1] (also see[P, G, GTZ2]).

Proposition 3.4. *The derived Hochschild chain $(X_\bullet, A) \mapsto CH_{X_\bullet}(A)$ lifts as a functor of $(\infty,1)$-categories*

$$CH : sSet_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg.$$

Further, it is the tensor of A and $X_\bullet$ in E_∞-Alg, i.e.*, there is an natural equivalence $CH_{X_\bullet}(A) \cong A \boxtimes X_\bullet$. In particular,*

$$Map_{sSet_\infty}\big(X_\bullet, Map_{E_\infty\text{-}Alg}(A,B)\big) \cong Map_{E_\infty\text{-}Alg}(CH_{X_\bullet}(A), B). \tag{6}$$

Note that the tensor definition $A \boxtimes X_\bullet$ could also be used to actually define higher Hochschild chains.

Proof. Proposition 3.6 below implies that the derived Hochschild chain functor is invariant under (weak) equivalences of E_∞-algebras and simplicial sets and thus lifts as an $((\infty,1)$-)functor $sSet_\infty \times E_\infty\text{-}Alg \to k\text{-Mod}_\infty$, $(X_\bullet, A) \mapsto CH_{X_\bullet}(A)$. Since the tensor products of E_∞-algebras is an E_∞-algebra, $CH_{K_\bullet}^{simp}(A)$ is a simplicial E_∞-algebra for any simplicial set $K_\bullet$. Since the (refined) Dold-Kan functor $sE_\infty\text{-}Alg \to E_\infty\text{-}Alg$ preserves weak-equivalences [M1], the derived Hochschild chain functor lifts as a functor of $(\infty,1)$-categories $CH : sSet_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$. By Proposition 2.7, $CH_{X_\bullet}(A) \cong A \boxtimes X_\bullet$ in $sE_\infty\text{-}Alg$ from which the second assertion of the Proposition follows after passing to geometric realization. ☐

[6]it is unique up to contractible choices

Remark 3.5. There is a derived functor interpretation of the above Definition 3.2. Recall that to any simplicial set $X_\bullet$ one can associate a canonical E_∞-coalgebra structure [Ma, BF], denoted $C_*(X_\bullet)$ (Example 2.5). Dually to the case of algebras, an E_∞-coalgebra C defines a *contravariant* monoidal functor $X \mapsto CH^{simp^{co}}_X(C)$, *i.e.*, an object of $Fun^{\otimes}(N(Fin)^{op}, k\text{-Mod}_\infty)$.

In particular, an E_∞-coalgebra C defines a *right* module over the ∞-operad $\mathbb{E}_\infty^{\otimes}$ and an E_∞-algebra a *left* module over the ∞-operad $\mathbb{E}_\infty^{\otimes}$. We can thus form their (derived) tensor products $C \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^{\otimes}}{\otimes}} A$ which is computed as a (homotopy) coequalizer:

$$C \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^{\otimes}}{\otimes}} A \cong hocoeq\Big(\coprod_{f:\{1,\dots,q\}\to\{1,\dots,p\}} C^{\otimes p} \otimes \mathbb{E}_\infty^{\otimes}(q,p) \otimes A^{\otimes q} \rightrightarrows \coprod_n C^{\otimes n} \otimes A^{\otimes n}\Big)$$

where the maps $f : \{1,\dots,q\} \to \{1,\dots,p\}$ are maps of sets. The upper map in the coequalizer is induced by the maps $f^* : C^{\otimes p} \otimes \mathbb{E}_\infty^{\otimes}(q,p) \otimes A^{\otimes q} \to C^{\otimes q} \otimes A^{\otimes q}$ obtained from the coalgebra structure of C and the lower map is induced by the maps $f_* : C^{\otimes p} \otimes \mathbb{E}_\infty^{\otimes}(q,p) \otimes A^{\otimes q} \to C^{\otimes p} \otimes A^{\otimes p}$ induced by the algebra structure.

Proposition 3.6. *Let $X_\bullet$ be a simplicial set and A be an E_∞-algebra. There is an natural equivalence*

$$\begin{aligned} CH_{X_\bullet}(A) \cong & C_*(X_\bullet) \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^{\otimes}}{\otimes}} A \\ \cong & hocoeq\Big(\coprod_{f:\{1,\dots,q\}\to\{1,\dots,p\}} C_*(X_\bullet)^{\otimes p} \otimes \mathbb{E}_\infty^{\otimes}(q,p) \otimes A^{\otimes q} \rightrightarrows \coprod_n C_*(X_\bullet)^{\otimes n} \otimes A^{\otimes n}\Big) \end{aligned}$$

Proof. Note that the E_∞-coalgebra structure on $C_*(X_\bullet)$ is given by the functor $N(Fin_*)^{op} \to k\text{-Mod}_\infty$ defined by $I \mapsto k\big[Hom_{Fin}(I, X_\bullet)\big]$. The rest of the proof now is the same as in [GTZ2, Proposition 4]. □

In [GTZ2], a functor $CH^{cdga} : sSet_\infty \times CDGA_\infty \to CDGA_\infty$ was defined[7]. There is a forgetful functor $CDGA_\infty \to E_\infty\text{-}Alg$. Proposition 3.6, Proposition 4 in [GTZ2] and the equivalence $\mathbb{E}_\infty^{\otimes} \overset{\simeq}{\to} Comm^{\otimes}$ yield

Corollary 3.7. *If A is a commutative differential graded algebra, then $CH^{cdga}_{X_\bullet}(A)$ is naturally equivalent to $CH_{X_\bullet}(A)$.*

Further, the following diagram is (homotopy) commutative in the (∞-)category $Fun(sSet_\infty \times CDGA_\infty, E_\infty\text{-}Alg)$:

$$\begin{tikzcd} sSet_\infty \times CDGA_\infty \arrow[r, "CH^{cdga}"] \arrow[d] & CDGA_\infty \arrow[d] \\ sSet_\infty \times E_\infty\text{-}Alg \arrow[r, "CH"] & E_\infty\text{-}Alg \end{tikzcd}$$

In other words, the corollary means that the functors CH^{cdga} and CH are equivalent (for a CDGA).

Remark 3.8. In the sequel we will use the equivalence given by corollary 3.7 to identify the functors CH and CH^{cdga} without further notice.

[7]it was simply denoted CH in loc. cit.

There is an equivalence of $(\infty,1)$-categories $sSet_\infty \underset{\sim}{\overset{\sim}{\rightleftarrows}} Top_\infty$ induced by the underlying Quillen equivalence in between $sSet$ and Top [GJ, Ho]. The left and right equivalences are respectively induced by the standard singular set functor $X \mapsto S_\bullet(X) := Map(\Delta^\bullet, X)$ and geometric realization $X_\bullet \mapsto |X_\bullet|$ functors. In particular, we can replace simplicial sets by *topological spaces* in Definition 3.2 and Proposition 3.6 to get the following analogue of Proposition 3.4. Letting $C_*(X)$ be the natural E_∞-coalgebra structure on the singular chains of X, we deduce from Proposition 3.4 and Proposition 3.6:

Proposition 3.9. *The derived* Hochschild chain with value in an E_∞-algebra A modeled on a space X *given by*

$$\begin{aligned} CH_X(A) &:= DK\Big(\varinjlim_{Fin \ni K \to S_\bullet(X)} CH_K^{simp}(A)\Big) \\ &\cong C_*(X) \underset{\mathbb{E}_\infty^{\otimes}}{\overset{\mathbb{L}}{\otimes}} A \end{aligned}$$

induces a $(\infty,1)$ functor $CH : (X_\bullet, A) \mapsto CH_{X_\bullet}(A)$ from $Top_\infty \times E_\infty$-$Alg$ to E_∞-Alg. Further, one has an natural equivalence $A \boxtimes X \cong CH_X(A)$.

Remark 3.10. Since $(X,A) \mapsto CH_X(A) \cong A \boxtimes X$ is a functor of both variables, $CH_X(A)$ has an natural action of the topological monoid $Map_{Top_\infty}(X,X)$ (and in particular of the group $Homeo(X)$), *i.e.*, there is a monoid map $Map_{Top_\infty}(X,X) \to Map_{E_\infty\text{-}Alg}(CH_X(A), CH_X(A))$; in other words a chain map $C_*\big(Map_{Top_\infty}(X,X)\big) \otimes CH_X(A) \to CH_X(A)$ which makes $CH_X(A)$ a $Map_{Top_\infty}(X,X)$-algebra in E_∞-Alg (for the monad associated to the monoid $Map_{Top_\infty}(X,X)$).

Similarly, given any map $f : X \times K \to Y$ of topological spaces, we get a canonical map $K \to Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Y(A))$ or, equivalently, a canonical map $f_* : C_*(K) \otimes CH_X(A) \to CH_Y(A)$ in k-Mod$_\infty$ (which is simply induced by $f_* : CH_{K\times X}(A) \to CH_Y(A)$ under the equivalence

$$Map_{Top_\infty}\big(K, Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Y(A))\big) \cong Map_{E_\infty\text{-}Alg}\big(CH_{X\times K}(A), CH_Y(A)\big)$$

given by Proposition 3.9 and Corollary 3.27.(4)).

As we previously mentioned, the higher Hochschild functor (modeled on spaces) agrees with factorization homology (see [L-HA, F1] and Definition 2.9) for E_∞-algebras. Indeed the following result (whose CDGA version was proved in [GTZ2]) was proved by Francis [F1].

Theorem 3.11. *Let M be a manifold of dimension m and A be an E_∞-algebra viewed as an $N(Disk(M))$-algebra (by restriction of structure, Example 2.12). Then, the factorization homology $\int_M A$ of M with coefficients in A is naturally equivalent to $CH_M(A)$.*

Proof. The proof is the same as the ones for CDGA's in [GTZ2] (see Theorem 6 and Corollary 9 in *loc. cit.*) using the axioms of Theorem 3.26. Further, as pointed out by John Francis [F1], the proof also applies to topological manifolds. $\square$

In particular, it follows that the factorization homology of an E_∞-algebra and framed manifold M is canonically an E_∞-algebra which is independent of the choices of framing, and further, can be extended functorially with respect to all continuous maps $h : N \to M$.

Remark 3.12. There is also a nice interpretation of Hochschild chains over spaces in terms of derived (or homotopical) algebraic geometry. Let $\mathbf{dSt}_k$ be the (model) category of derived stacks over the ground ring k described in details in [TV, Section 2.2]. This category admits internal Hom's that we denote by $\mathbb{R}\,Map(F,G)$ following [TV, TV2] and further is also an enrichment of the homotopy category of spaces. Indeed, any simplicial set $X_\bullet$ yields a constant simplicial presheaf $E_\infty\text{-}Alg \to sSet$ defined by $R \mapsto X_\bullet$ which, in turn, can be stackified. We denote $\mathfrak{X}$ the associated stack, *i.e.* the stackification of $R \mapsto X_\bullet$, which depends only on the (weak) homotopy type of $X_\bullet$. For a (derived) stack $\mathfrak{Y} \in \mathbf{dSt}_k$, we denote $\mathcal{O}_{\mathfrak{Y}}$ its functions [TV] (*i.e.*, $\mathcal{O}_{\mathfrak{Y}} := \mathbb{R}\underline{Hom}(\mathfrak{Y}, \mathbb{A}^1)$).

Corollary 3.13. *Let $\mathfrak{R} = \mathbb{R}\,Spec(R)$ be an affine derived stack (for instance an affine stack)* [TV] *and $\mathfrak{X}$ be the stack associated to a space X. Then the Hochschild chains over X with coefficient in R represent the mapping stack $\mathbb{R}\,Map(\mathfrak{X},\mathfrak{R})$. That is, there are canonical equivalences*

$$\mathcal{O}_{\mathbb{R}\,Map(\mathfrak{X},\mathfrak{R})} \;\cong\; CH_X(R), \qquad \mathbb{R}\,Map(\mathfrak{X},\mathfrak{R}) \;\cong\; \mathbb{R}\,Spec\big(CH_X(R)\big)$$

Proof. The proof is analogous to the one of [GTZ2, Corollary 6.4.4]. □

Note that if a group G acts on X, the natural action of G on $CH_X(A)$ (see Remark 3.10) identifies with the natural of G on $\mathbb{R}\,Map(\mathfrak{X},\mathfrak{R})$ under the equivalence given by Corollary 3.13.

3.2. Higher Hochschild (co)chains with values in E_∞-modules.

We now consider a dual notion of the Hochschild chain functor, which is well defined in the E_∞-case.

Let $\epsilon : pt \to X_\bullet$ be a base point of $X_\bullet$. The map ϵ yields a map of E_∞-algebras $A \cong CH_{pt}(A) \stackrel{\epsilon_*}{\to} CH_{X_\bullet}(A)$ and thus makes $CH_{X_\bullet}(A)$ an A-module. Let M be another E_∞-A-module.

Definition 3.14. The (derived) *Hochschild cochains* of an E_∞-algebra A with value in M over (the pointed simplicial set) $X_\bullet$ is given by

$$CH^{X_\bullet}(A,M) = Hom_A(CH_{X_\bullet}(A), M),$$

the (derived) chain complex of the underlying left E_1-A-modules homomorphisms.

The definition above depends on the choice of the base point even though we do not write it explicitly in the definition. We define similarly $CH^X(A,M)$ for any *pointed topological space* X.

Remark 3.15. According to Theorem 6.6, on can also alternatively consider the chain complex of E_∞-A-modules in Definition 3.14.

Definition 3.16. The *Hochschild chains* of an E_∞-algebra A with values in M over (the pointed simplicial set) $X_\bullet$ is defined as

$$CH_{X_\bullet}(A,M) = M \overset{\mathbb{L}}{\underset{A}{\otimes}} CH_{X_\bullet}(A)$$

the relative tensor products of E_∞-A-modules (as defined, for instance, in [L-HA, Section 3.3.3] or [KM]).

Remark 3.17. Any E_∞-A-module has an underlying E_1-module structure given by the forgetful functor $A\text{-}Mod^{E_\infty} \to A\text{-}Mod^{E_1}$ hence both a left and right A-module structure. Thus, given two E_∞-A-modules M, N, one can form their relative tensor product $M \otimes^{\mathbb{L}}_A N$ where M is viewed as a right A-module, N as a left A-module and A as an E_1-algebra. According to Theorem 6.6 and [L-HA, Section 4.4.1] or [KM, Section 5], this tensor is equivalent (as an object of k-Mod_∞) to the relative tensor product computed in E_∞-A-modules. Hence, the tensor product of Definition 3.16 can be computed using this alternative definition.

Since the based point map $\epsilon_* : A \to CH_{X_\bullet}(A)$ is a map of E_∞-algebras, the canonical module structure of $CH_{X_\bullet}(A)$ over itself induces a structure of module on $CH_{X_\bullet}(A, M)$ after tensoring by A (also see [KM, Part V], [L-HA]):

Lemma 3.18. *Let M be in $A\text{-}Mod^{E_\infty}$. Then $CH_{X_\bullet}(A, M)$ is canonically a $CH_{X_\bullet}(A)$-E_∞-module.*

Remark 3.19. By definition, if A is endowed with its canonical A-E_∞-module structure, the natural map $CH_{X_\bullet}(A, A) \cong A \otimes^{\mathbb{L}}_A CH_{X_\bullet}(A) \to CH_{X_\bullet}(A)$ is an equivalence of $CH_{X_\bullet}(A)$-modules. Hence, tensoring by $M \otimes_A -$, we get a canonical lift of the relative tensor products $M \otimes^{\mathbb{L}}_A CH_{X_\bullet}(A)$, computed as a relative tensor product of left and right modules over A seen as an E_1-algebra, to an $CH_{X_\bullet}(A)$-E_∞-module as well.

Proposition 3.20. *The derived Hochschild chain $CH_{X_\bullet}(A, M)$ with value in an E_∞-algebra A and an A-module M over a space $X_\bullet$ given by Definition 3.16 induces a functor of $(\infty,1)$-categories $CH : (X_\bullet, M) \mapsto CH_{X_\bullet}(\iota(M), M)$ from $sSet_{*\infty} \times Mod^{E_\infty}$ to Mod^{E_∞}.*

The derived Hochschild cochains $CH^{X_\bullet}(A, M)$ with value in an E_∞-algebra A and an A-module M over a space $X_\bullet$ given by Definition 3.14 induces a functor of $(\infty,1)$-categories $(X_\bullet, M) \mapsto CH^{X_\bullet}(A, M)$ from $sSet_{*\infty}^{op} \times A\text{-}Mod^{E_\infty}$ to $A\text{-}Mod^{E_\infty}$, which is further contravariant[8] with respect to A.

Proof. It follows from Lemma 3.18 and § 3.1. The fact that homomorphisms of A-E_∞-modules have a canonical structure of A-E_∞-modules follows from the same argument as for the tensor product above or from [KM, Theorem V.8.1]. ☐

Remark 3.21. As usual, one obtains a similar version of the above Definition 3.16 and Lemma 3.18 for pointed topological space X.

Remark 3.22. If A is a CDGA and M a left A-module, similarly to Corollary 3.7, there are natural equivalences

$$CH^{cdga}_{X_\bullet}(A, M) \cong CH_{X_\bullet}(A, M), \qquad CH^{X_\bullet}_{cdga}(A, M) \cong CH^{X_\bullet}(A, M)$$

where $CH^{cdga}_{X_\bullet}(A, M)$ and $CH^{X_\bullet}_{cdga}(A, M)$ are the usual higher Hochschild chains and cochains functors for CDGA's and their modules defined respectively in [P] and [G].

[8] using the canonical functor (similar to the one of Example 2.2) $f_* : B\text{-}Mod^{E_\infty} \to A\text{-}Mod^{E_\infty}$ associated to any E_∞-algebras map $f : A \to B$

3.3. **Axiomatic characterization.** The axiomatic approach to Hochschild functors over spaces for CDGA's studied in the authors previous work [GTZ2] extends formally to E_∞-algebras as well. It is actually an immediate corollary of the fact that E_∞-Alg (as well as any presentable $(\infty,1)$-category) is tensored over simplicial sets in an unique way (up to homotopy). We now recall quickly the axiomatic characterization (similar to the Eilenberg-Steenrod axioms) and some consequences for Hochschild theory over spaces with value in Mod^{E_∞}. A similar story for factorization homology of E_n-algebras has been developed recently by Francis [F2, AFT].

We first collect the axioms characterizing the (derived) Hochschild chain theory over spaces into the following definition. Let $Forget : Top_{*\infty} \to Top_\infty$ be the functor that forget the base point.

Definition 3.23. An E_∞-homology theory[9] is a pair of ∞-functors $\mathcal{CA} : Top_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$, denoted $(X,A) \mapsto \mathcal{CA}_X(A)$, and $\mathcal{CM} : Top_{*\infty} \times Mod^{E_\infty} \to Mod^{E_\infty}$, denoted $(X,M) \mapsto \mathcal{CM}_X(M)$, fitting in a commutative diagram

$$\begin{tikzcd} Top_{*\infty} \times Mod^{E_\infty} \arrow[r, "\mathcal{CM}"] \arrow[d, "Forget\times\iota"'] & Mod^{E_\infty} \arrow[d, "\iota"] \\ Top_\infty \times E_\infty\text{-}Alg \arrow[r, "\mathcal{CA}"] & E_\infty\text{-}Alg \end{tikzcd} \tag{7}$$

satisfying the following axioms:

i) value on a point: there is a natural equivalence $\mathcal{CM}_{pt}(M) \cong M$ in Mod^{E_∞};

ii) monoidal: there is a natural equivalence

$$\mathcal{CM}_{X\coprod Y}(M) \cong \mathcal{CM}_X(M) \otimes \mathcal{CA}_Y(\iota(M))$$

(where $X \in Top_{*\infty}$ and $Y \in Top_\infty$),

iii) excision: $\mathcal{CM}$ commutes with homotopy pushout of spaces, *i.e.*, there is a natural equivalence

$$\mathcal{CM}_{X\cup^h_Z Y}(M) \cong \mathcal{CM}_X(M) \overset{\mathbb{L}}{\underset{\mathcal{CA}_Z(\iota(M))}{\otimes}} \mathcal{CA}_Y(\iota(M))$$

where $X \in Top_{*\infty}$, $Y,Z \in Top_\infty$.

Remark 3.24. Since any E_∞-algebra is canonically a module over itself, there is also a canonical functor $\phi : E_\infty\text{-}Alg \to Mod^{E_\infty}$, hence a functor $\big(-\coprod\{*\}\big)\times\phi : Top_\infty \times E_\infty\text{-}Alg \to Top_{*\infty} \times Mod^{E_\infty}$ giving rise, by composition with $\iota\circ\mathcal{CM}$ to a functor $\psi : Top_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$. By axioms *i*) and *ii*) in Definition 3.23 and commutativity of the diagram (7), we get a natural equivalence

$$\psi_X(A) \;\cong\; \phi(A)\otimes\mathcal{CA}_X(A).$$

Hence, the functor $\mathcal{CA}$ is actually completely defined by the functor $\mathcal{CM}$.

Remark 3.25. We also define a generalized E_∞-homology theory to be a triple of functors $F : Mod^{E_\infty} \to Mod^{E_\infty}$, $\mathcal{CA} : Top_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$ and $\mathcal{CM} : Top_{*\infty} \times Mod^{E_\infty} \to Mod^{E_\infty}$ satisfying all properties as in Definition 3.23 except that the value on a point axiom is modified by requiring a natural equivalence $\mathcal{CM}_{pt}(M) \cong F(M)$ in Mod^{E_∞}.

[9] with value in the symmetric monoidal $(\infty,1)$-category $(k\text{-Mod}_\infty,\otimes)$

The next Theorem shows that higher Hochschild homology theory is the unique functor satisfying the assumptions of Definition 3.23.

Theorem 3.26. (1) *The derived Hochschild chains functors $CH : Top_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$ (see Proposition 3.9) and the derived Hochschild chains with value in a module $CH_X : Top_{*\infty} \times Mod^{E_\infty} \to Mod^{E_\infty}$ (see Proposition 3.20) is a E_∞-homology theory in the sense of Definition 3.23.*

(2) *Any E_∞-homology theory (in the sense of Definition 3.23) is naturally equivalent to derived Hochschild chains,* i.e.*, there are natural equivalences $\mathcal{CA}_X(A) \cong CH_X(A)$ and $\mathcal{CM}_X(M) \cong CH_X(\iota(M), M)$.*

Proof. This is essentially implied by the fact that $CH_X(A) \cong A \boxtimes X$ is the tensor of A with the space X and that such a tensor is defined uniquely, see [L-HTT, Corollary 4.4.4.9]. Note that the first assertion follows from Proposition 3.20 and Proposition 3.9. The proof of the uniqueness follows from the proofs of Theorem 4.2.7 and Theorem 4.3.1 in [GTZ2]. The excision and the value on a point axioms applied to $X = Z = pt$ show that there is a natural equivalence

$$\mathcal{CM}_Y(M) \cong M \overset{\mathbb{L}}{\underset{\iota(M)}{\otimes}} \mathcal{CA}_Y(\iota(M))$$

which reduces to proving the assertion for $\mathcal{CA}$. Since $\iota : Mod^{E_\infty} \to E_\infty\text{-}Alg$ is monoidal, $\mathcal{CA}$ is monoidal. Similarly, the natural equivalence(4) implies that $\mathcal{CA}$ satisfies the excision axiom (in the category of E_∞-algebras). Now the proof of [GTZ2, Theorem 2] applies verbatim. The argument boils down to the fact that Top_∞ is generated by a point using coproducts and homotopy pushouts. □

We now list a few easy properties derived from the above Theorem 3.26.

Corollary 3.27. (1) *The derived Hochschild chain functor is the unique functor $Top_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$ satisfying the following three axioms*

(a) **value on a point:** *There is a natural equivalence of E_∞-algebras $CH_{pt}(A) \cong A$.*

(b) **coproduct:** *There are natural equivalences*

$$CH_{\coprod_I (X_i)}(A) \cong \varinjlim_{\substack{K \subset I \\ K \text{ finite}}} \bigotimes_{k \in K} CH_{X_k}(A)$$

(c) **homotopy glueing/pushout:** *there are natural equivalences*

$$CH_{X \cup_Z^h Y}(A) \cong CH_X(A) \otimes^{\mathbb{L}}_{CH_Z(A)} CH_Y(A).$$

(2) ***(generalized uniqueness)*** *Let $F : Mod^{E_\infty} \to Mod^{E_\infty}$, $\mathcal{CA} : Top_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$ and $\mathcal{CM} : Top_{*\infty} \times Mod^{E_\infty} \to Mod^{E_\infty}$ be a generalized E_∞-homology theory. Then there is a natural equivalence*

$$\mathcal{CM}_X(M) \;\cong\; CH_X\big(\iota(F(M)), F(M)\big).$$

(3) ***(commutations with colimits)*** *The derived Hochschild chains functors $CH : Top_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$ and $CH : Top_{*\infty} \times Mod^{E_\infty} \to Mod^{E_\infty}$ commutes with finite colimits in Top_∞ and all colimits in Mod^{E_∞}, that is there are natural equivalences*

$$CH_{\varinjlim_{\mathcal{F}} X_i}(\iota(M), M) \cong \varinjlim_{\mathcal{F}} CH_{X_i}(\iota(M), M) \quad (\textit{for a finite category } \mathcal{F}),$$

$$CH_X(\varinjlim A_i) \cong \varinjlim CH_X(A_i).$$

(4) **(product)** *Let X, Y be pointed spaces, $M \in Mod^{E_\infty}$ and $A = \iota(M) \in E_\infty$-$Alg$. There is a natural equivalence*

$$CH_{X\times Y}(A, M) \xrightarrow{\simeq} CH_X\left(CH_Y(A), CH_Y(A, M)\right))$$

in Mod^{E_∞}.

Proof. The proof of the first assertion follows directly from Theorem 3.26 by applying the monoidal functor ι. The proof of the other assertions are the same as the analogous result for CDGA's proved in [GTZ2]. ☐

4. E_∞ Chen models for mapping spaces

This section is devoted to the relationship in between higher Hochschild chains and mapping spaces. In particular, we prove an E_∞-algebra version of the Chen iterated integral morphism studied in [GTZ].

Let A be an E_∞-algebra. Recall that by the coproduct axiom and functoriality of Hochschild chains (see Theorem 3.26, Corollary 3.27), there is a natural equivalence $A \otimes A \cong CH_{S^0}(A)$ of E_∞-algebras as well as a natural E_∞-algebras map $CH_{S^0}(A) \to CH_{pt}(A) \cong A$.

Lemma 4.1. *Let X, Y be topological spaces and $C^*(X)$, $C^*(Y)$ be their E_∞-algebras of cochains. Denote $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ the projections onto the first and second factors. The composition,*

$$(8)\quad C^*(X)\otimes C^*(Y) \stackrel{\pi_X^*\otimes\pi_Y^*}{\longrightarrow} C^*(X\times Y)\otimes C^*(X\times Y) \xrightarrow{\simeq} CH_{S^0}\big(C^*(X\times Y)\big) \longrightarrow CH_{pt}\big(C^*(X\times Y)\big) \cong C^*(X\times Y)$$

is a natural equivalence of E_∞-algebras.

Proof. That the maps involved are natural (in $X, Y \in Top_\infty$) maps of E_∞-algebras follows from the functoriality of $X \mapsto C^*(X)$ and the functorial and monoidal properties of the higher Hochschild derived functor (see Theorem 3.26).

We now prove that the map (8) is an equivalence. Note that if the ground ring k is a field (or if the cohomology groups of X, Y are torsion free), the map (8) induces a map $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ which is canonically identified with the Künneth isomorphism since for a graded commutative algebra, the map $A \otimes A \cong CH_{S^0}(A) \to CH_{pt}(A) \cong A$ is given by the multiplication in A (by Corollary 3.7).

For a general ground ring coefficient, note that as a mere E_1-algebra (via the forgetful functor E_∞-$Alg \hookrightarrow E_1$-Alg), the singular cochain complex $C^*(X)$ is endowed with the (strictly) associative algebra structure given by the cup-product. Let D^1_+, D^1_- be two open disjoint sub-intervals of D^1 and $i : D^1_- \coprod D^1_+ \hookrightarrow D^1$ be the inclusion map. By definition (see [L-HA, Lu3, F1]), for any differential graded associative algebra (A, m), the canonical map of *chain complexes* (and not E_1-algebras)

$$A\otimes A \;\cong\; \int_{D^1_-\coprod D^1_+} A \xrightarrow{i_*} \int_{D^1} A \;\cong\; A$$

is the multiplication map $m : A \otimes A \to A$ defining the E_1-structure of A. If furthermore (A, m) is actually an E_∞-algebra, by Theorem 3.11 and functoriality

of derived Hochschild functor, there is a (homotopy) commutative diagram of chain complexes

$$\begin{tikzcd}
A\otimes A \arrow[ddd, "m"'] \arrow[dr, "\simeq"'] \arrow[drr, bend left=10, "\simeq"] \arrow[drrr, bend left=15, "\simeq"] & & & \\
& \int_{D^1_-\coprod D^1_+} A \arrow[r, "\simeq"] \arrow[d, "i"'] & CH_{D^1_-\coprod D^1_+}(A) \arrow[r, "\simeq"] \arrow[d, "i_*"] & CH_{S^0}(A) \arrow[d] \\
& \int_{D^1} A \arrow[r, "\simeq"] \arrow[dl, "\simeq"] & CH_{D^1}(A) \arrow[r, "\simeq"] \arrow[dll, bend left=10, "\simeq"] & CH_{pt}(A) \arrow[dlll, bend left=15, "\simeq"] \\
A & & &
\end{tikzcd}$$

and thus, the map (8) is homotopy equivalent, as a map of chain complexes, to

$$C^*(X)\otimes C^*(Y) \xrightarrow{\pi_X^*\otimes\pi_Y^*} C^*(X\times Y)\otimes C^*(X\times Y) \xrightarrow{\cup} C^*(X\times Y). \tag{9}$$

The cochain complex structure of $C^*(X)$ is the normalization of the cosimplicial k-module $n\mapsto C^n(X)$ so that the above map (9) is the Alexander-Whitney diagonal $AW: C^*(X)\otimes C^*(Y)\to C^*(X\times Y)$ (in k-Mod$_\infty$). Since the Alexander-Whitney map is a quasi-isomorphism, the lemma follows. □

Remark 4.2. The map of E_∞-algebra $C^*(X)\otimes C^*(Y)\to C^*(X\times Y)$ given by Lemma 4.1 is in particular a map of chain complexes. From the last part of the proof of Lemma 4.1, it follows that this map is equivalent in k-Mod_∞ to the Alexander-Whitney diagonal (see the map (9)), *i.e.* the map given by Lemma 4.1 is an E_∞-lifting of the Alexander-Whitney diagonal.

Let $X_\bullet$ be a simplicial set and Y be a topological space. We define a map

$$ev: Y^{|X_\bullet|}\times\Delta^n\to Y^{X_n}$$

by $ev(f,(t_0,\cdots,t_n))=g$, where for

$$f:\Big(\coprod(X_n\times\Delta^n)/\sim\Big)\to Y \text{ and } (t_0,\cdots,t_n)\in\Delta^n,$$

we have,

$$g(\sigma_n)=f([\sigma_n,(t_0,\cdots,t_n)]).$$

Note that this is a well defined map of cosimplicial topological spaces. In fact, ev is induced by the canonical map $X_n\to Map(\Delta^n,|X_\bullet|)$ given by the unit of the adjunction between simplicial sets and topological spaces.

Applying the E_∞ cochain functor $C^*(-)$ (Example 2.5) yields a natural map

$$ev^*:\big(C^*(Y^{X_i})\big)_{(i\in\mathbb{N})}\to\big(C^*(Y^{|X_\bullet|}\times\Delta^i)\big)_{(i\in\mathbb{N})} \tag{10}$$

of simplicial E_∞-algebras.

Lemma 4.3. *The geometric realization of the simplicial E_∞-algebra $\big(C^*(Z)\times\Delta^i)\big)_{(i\in\mathbb{N})}$ is naturally equivalent to $C^*(Z)$, as an E_∞-algebra.*

Proof. By Lemma 4.1, there is a natural equivalence $C^*(Z\times\Delta^i)\cong C^*(Z)\otimes C^*(\Delta^i)$ in E_∞-*Alg*. This induces an equivalence,

$$C^*(Z)\otimes\big(C^*(\Delta^i)\big)_{(i\in\mathbb{N})}\xrightarrow{\simeq}\big(C^*(Z)\times\Delta^i)\big)_{(i\in\mathbb{N})}$$

of simplicial E_∞-algebras. Since the constant map $\Delta^i\to pt$ is a homotopy equivalence, the canonical map $C^*(pt)\to C^*(\Delta^i)$, where $C^*(pt)$ is viewed as a constant simplicial E_∞-algebra, is an equivalence. Composing the above with the equivalence,

$$C^*(Z)\otimes\big(C^*(pt)\big)_{(i\in\mathbb{N})}\xrightarrow{\simeq}C^*(Z)\otimes\big(C^*(\Delta^i)\big)_{(i\in\mathbb{N})}$$

gives rise to an equivalence between $C^*(Z)$ and the constant simplicial E_∞-algebra $C^*(Z\times\Delta^i)$. ☐

Let $X_\bullet$ be a simplicial set. Iterating Lemma 4.1, we get, for any $n\in\mathbb{N}$, a natural map of E_∞-algebras

$$CH_{X_n}(C^*(Y))\longrightarrow C^*\big(Y^{X_n}\big) \tag{11}$$

Composing the map (11) with the ev^* map in (10), we get a natural morphism of simplicial E_∞-algebras,

$$\mathcal{I}t: CH^{simp}_{X_\bullet}(C^*(Y))\longrightarrow C^*\big(Y^{X_\bullet}\big)\xrightarrow{ev^*}C^*\big(Y^{|X_\bullet|}\times\Delta^\bullet\big). \tag{12}$$

The following result is an integral, E_∞-lifting of the iterated integrals [GTZ2].

Theorem 4.4. *The realization of the map $\mathcal{I}t: CH^{simp}_{X_\bullet}(C^*(Y))\to C^*\big(Y^{|X_\bullet|}\times\Delta^\bullet\big)$ yields a natural morphism of E_∞-algebras*

$$\mathcal{I}t: CH_{X_\bullet}(C^*(Y))\longrightarrow C^*\big(Y^{|X_\bullet|}\big).$$

*Further, if $|X_\bullet|$ is n-dimensional (*i.e.* the highest degree of any non-degenerate simplex is n) and Y is n-connected, then the map $\mathcal{I}t$ is an equivalence.*

Proof. Since the natural map (11), $CH^{simp}_{X_\bullet}(C^*(Y))\to C^*\big(Y^{X_\bullet}\big)$, and the map (10), $C^*\big(Y^{X_\bullet}\big)\overset{ev^*}{\to}C^*\big(Y^{|X_\bullet|}\times\Delta^\bullet\big)$, are simplicial, their realization yields a map of E_∞-algebras

$$CH_{X_\bullet}(C^*(Y))\to C^*\big(Y^{|X_\bullet|}\big)\ \cong\ C^*\big(Y^{|X_\bullet|}\big)$$

where the last equivalence follows from Lemma 4.3. This defines the map $\mathcal{I}t$ which is natural by construction.

Now, we assume $|X_\bullet|$ is n-dimensional and Y is n-connected. We only need to check that the underlying map of *cochain complexes* $CH_{X_\bullet}(C^*(Y))\to C^*\big(Y^{|X_\bullet|}\times\Delta^\bullet\big)$ is an equivalence in the $(\infty,1)$-category of cochain complexes. The proof of Lemma 4.1 (see Remark (4.2)) implies that the cochain complex morphism

$$CH_{X_\bullet}(C^*(Y))\longrightarrow C^*(Y^{X_\bullet})$$

is the map induced by the iterated Alexander-Whitney diagonal. Since the geometric realization commutes with the forgetful functor E_∞-$Alg\to k$-Mod_∞, the geometric realization of the map $C^*(Y^{X_\bullet})\to C^*\big(Y^{|X_\bullet|}\times\Delta^\bullet\big)$ is equivalent in k-Mod_∞ to the map induced by the slant products

$$C^\ell(Y^{X_n})\to C^\ell\Big(Y^{|X_\bullet|}\times\Delta^\ell\Big)\xrightarrow{/[\Delta^n]}C^{\ell-n}\Big(Y^{|X_\bullet|}\Big)$$

by the fundamental chain $[\Delta^n]$ given by the unique non-degenerate n-simplex of Δ^n.

Hence we have proved that $\mathcal{I}t$ is equivalent in k-Mod_∞ to the composition

$$\bigoplus_{n\geq 0} CH_{X_n}(C^*(Y)) \longrightarrow \bigoplus_{n\geq 0} C^*(Y^{X_n})$$
$$\xrightarrow{\bigoplus ev^*} \bigoplus_{n\geq 0} C^*\Big(Y^{|X_\bullet|}\times \Delta^n\Big) \xrightarrow{\bigoplus_{n\geq 0}/[\Delta^n]} C^*\Big(Y^{|X_\bullet|}\Big).$$

This last map is a quasi-isomorphism under the appropriate assumptions on $X_\bullet$ and Y using the same argument as in [GTZ, PT]. □

Remark 4.5. Let $Y = M$ be a manifold and k a field of characteristic zero. Then, by Corollary (3.7) and homotopy invariance of higher Hochschild cochains, there is a natural equivalence of E_∞-algebras

$$CH_{X_\bullet}(C^*(M)) \;\cong\; CH^{cdga}_{X_\bullet}(\Omega(M)).$$

Unfolding the proof of Theorem 4.4 and the construction of the map in Lemma 4.1, one can check that the map $\mathcal{I}t : CH_{X_\bullet}(C^*(M)) \to C^*(M^{|X_\bullet|})$, given by Theorem 4.4, coincides with the generalized Chen's iterated integrals defined in [GTZ, Section 2]. In particular, when $X_\bullet$ is the standard simplicial set model of the compact interval or the circle, we recover the original Chen iterated integral construction [Ch]. This justifies our notation $\mathcal{I}t$ for the map defined in Theorem 4.4.

Similarly, the argument of the proofs of Theorem 4.4 and Lemma 4.1 as well Theorem 3.11 (applied to the forgetful functor from E_∞-algebras to E_1-algebras) show that the iterated integral map $\mathcal{I}t : CH_{X_\bullet}(C^*(Y)) \to C^*\Big(Y^{|X_\bullet|}\Big)$ given by Theorem 4.4 is homotopy equivalent to the map of differential graded algebras described in [PT]. In particular, for $X = S^1_\bullet$, we recover an E_∞-algebra lift of Jones quasi-isomorphism [Jo].

Similarly, if X is a topological space, by choosing a simplicial model $X_\bullet$ for X (that is a simplicial set with an equivalence $|X_\bullet| \to X$), we get a natural equivalence $CH_X(C^*(Y)) \xrightarrow{\simeq} CH_{X_\bullet}(C^*(Y))$ and thus Theorem 4.4 yields the following corollary.

Corollary 4.6. *The map*

$$\mathcal{I}t : CH_X(C^*(Y)) \xrightarrow{\simeq} CH_{X_\bullet}(C^*(Y)) \longrightarrow C^*(Y^X)$$

is a natural morphism of E_∞-algebras and an equivalence if Y is $\dim(X)$-connected.

We will give a cohomological version of Theorem 4.4. Assume now that X is pointed (by a map $\epsilon : pt \to X$) and choose a pointed simplicial set model $X_\bullet$ of X. By naturality of the map $\mathcal{I}t$ in Theorem 4.4, there is a commutative diagram of E_∞-algebras maps:

$$\begin{tikzcd} CH_{X_\bullet}(C^*(Y)) \arrow[r, "\mathcal{I}t"] & C^*\big(Y^{|X_\bullet|}\big) \\ C^*(Y)\cong CH_{pt}(C^*(Y)) \arrow[u, "\epsilon_*"] \arrow[r, "\mathcal{I}t"] & C^*\big(Y^{pt}\big)\cong C^*(Y). \arrow[u, "C^*(\epsilon^*)"'] \end{tikzcd} \tag{13}$$

in which the lower map is the identity map by construction. It follows that $\mathcal{I}t$ is a $C^*(Y)$-E_∞-module map. Denoting $M^\vee = Hom_k(M,k)$ the linear dual of M (equipped with its canonical A-E_∞-structure if M is an A^{op}-E_∞-module), we thus get a map

$$\begin{aligned}\mathcal{I}t^* : C_*\Big(Y^X\Big) \cong C_*\Big(Y^{|X_\bullet|}\Big) &\longrightarrow Hom_k\Big(\big(C^*(Y^{|X_\bullet|})\big), k\Big) \\ &\xrightarrow{\simeq} Hom_{C^*(Y)}\Big(\big(C^*(Y^{|X_\bullet|})\big), \Big(C^*(Y)\Big)^\vee\Big) \\ &\xrightarrow{-\circ\mathcal{I}t} Hom_{C^*(Y)}\Big(CH_{X_\bullet}(C^*(Y)), \Big(C^*(Y)\Big)^\vee\Big) \\ &\cong CH^{X_\bullet}\Big(C^*(Y), (C^*(Y))^\vee\Big) \cong CH^X\Big(C^*(Y), (C^*(Y))^\vee\Big)\end{aligned} \tag{14}$$

where the first map is biduality morphism, the second map is the canonical isomorphism and the last two isomorphisms are from Definition 3.16.

Corollary 4.7. *The morphism $\mathcal{I}t^* : C_*\Big(Y^X\Big) \longrightarrow CH^X\Big(C^*(Y), (C^*(Y))^\vee\Big)$ in k-Mod_∞ is natural in X and Y.*

Further, if Y is $\dim(X)$-connected and the homology groups of Y are projective and finitely generated in each degree, then $\mathcal{I}t^*$ is a quasi-isomorphism.

Proof. That $\mathcal{I}t^*$ is natural in X and Y is immediate since all maps involved in its definition are natural in their two arguments. The assumption Y is $\dim(X)$-connected ensures that $\mathcal{I}t$ is a quasi-isomorphism. Further the above assumption together with the assumption on the homology groups of Y ensures that the biduality map $C_*\Big(Y^{|X_\bullet|}\Big) \longrightarrow Hom_k\big((C^*(Y^{|X_\bullet|})), k\big)$ is a quasi-isomorphism as well. $\square$

5. Algebraic structure of higher Hochschild cochains

5.1. **Wedge and cup products.** Let A be an E_∞-algebra and assume B is an A-algebra, *i.e.*, an E_∞-algebra object in the symmetric monoidal $(\infty,1)$-category A-Mod^{E_∞} of A-modules, see [L-HA, KM] for details.

Example 5.1. A map $f : A \to B$ of E_∞-algebras induces a natural E_∞-A-algebra structure on B.

Note further that, if B is a unital E_∞-A-algebra, then the map $a \mapsto a \cdot 1_B$ lifts to a map $f : A \to B$ of E_∞-algebras such that the induced E_∞-A-algebra structure on B is equivalent to the original one.

Since there is a canonical map $m_A : A\otimes A \to A$ of E_∞-algebras (Proposition 2.8), any A-module has a canonical structure of $A \otimes A$-module.

Lemma 5.2. *Let $M \in A$-Mod^{E_∞} be an A-module and X, Y be pointed topological spaces. There is a natural equivalence*

$$\mu : Hom_{A\otimes A}\left(CH_X(A) \otimes CH_Y(A), M\right) \xrightarrow{\simeq} CH^{X\vee Y}(A, M)$$

Proof. The excision property yields a natural equivalence

$$CH_{X\vee Y}(A) \cong A \overset{\mathbb{L}}{\underset{A\otimes A}{\otimes}} \Big(CH_X(A) \otimes CH_Y(A)\Big)$$

It follows that we have an equivalence

$$Hom_{A\otimes A}\left(CH_X(A) \otimes CH_Y(A), M\right) \;\cong\; Hom_A\left(CH_{X\vee Y}(A), M\right)$$

and the result now follows by Definition 3.14. $\square$

Using the above Lemma 5.2, for pointed spaces X, Y and B an A-algebra, we can define the following map

(15)
$$\mu_\vee : CH^X(A,B)\otimes CH^Y(A,B) \longrightarrow Hom_{A\otimes A}\Big(CH_X(A)\otimes CH_Y(A), B\otimes B\Big)$$
$$\stackrel{(m_B)_*}{\longrightarrow} Hom_{A\otimes A}\Big(CH_X(A)\otimes CH_Y(A), B\Big) \cong CH^{X\vee Y}(A,B)$$

where the first map is given by the tensor products $(f,g)\mapsto f\otimes g$ of functions.

Definition 5.3. We call $\mu_\vee : CH^X(A,B)\otimes CH^Y(A,B) \to CH^{X\vee Y}(A,B)$ the *wedge product* of Hochschild cochains (here we do not require that B is unital).

Note that this construction was already studied in some particular cases in our previous papers [G, GTZ].

Example 5.4. If A, B are actually CDGA's and given finite pointed set models $X_\bullet, Y_\bullet$ of X, Y, the map $\mu_\vee$ can be combinatorially described as follows. We have two cosimplicial chain complexes $CH^{X_\bullet}(A,B)\otimes CH^{Y_\bullet}(A,B)$ (with the diagonal cosimplicial structure) and $CH^{X_\bullet\vee Y_\bullet}(A,B)$. There is a cosimplicial map $\tilde{\mu}$: $CH^{X_\bullet}(A,B)\otimes CH^{Y_\bullet}(A,B)\to CH^{X_\bullet\vee Y_\bullet}(A,B)$ given, for any $f\in CH^{X_n}(A,B)\cong Hom_A(A^{\otimes \# X_n}, B)$, $g\in CH^{X_n}(A,B)\cong Hom_A(A^{\otimes\# Y_n},B))$ by

$$\mu(f,g)(a_0,a_2,\dots a_{\# X_n}, b_2,\dots,b_{\# Y_n}) = \pm a_0.f(1,a_2,\dots a_{\# X_n}).g(1,b_2,\dots,b_{\# Y_n})$$

where a_0 corresponds to the element indexed by the base point of $X_n\vee Y_n$ (the sign is given by the usual Koszul-Quillen sign convention). Composing the map $\tilde{\mu}$ with the Eilenberg-Zilber quasi-isomorphism realizes the wedge map (15):

$$\mu_\vee : CH^X(A,B)\otimes CH^Y(A,B)\to CH^{X\vee Y}(A,B).$$

Proposition 5.5. *The map $\mu_\vee$ is associative,* i.e.*, there is a commutative diagram*

$$\begin{tikzcd}
CH^X(A,B)\otimes CH^Y(A,B)\otimes CH^Z(A,B) \arrow[r, "\mu_\vee\otimes id"] \arrow[d, "id\otimes\mu_\vee"'] & CH^{X\vee Y}(A,B)\otimes CH^Z(A,B) \arrow[d, "\mu_\vee"] \\
CH^X(A,B)\otimes CH^{Y\vee Z}(A,B) \arrow[r, "\mu_\vee"] & CH^{X\vee Y\vee Z}(A,B)
\end{tikzcd}$$

in k-Mod_∞.

Proof. It follows from the associativity of the wedge product of spaces and tensor products of E_∞-algebras as used in Lemma 5.2 and Proposition 2.8. $\square$

Let X be a homotopy coassociative co-H-space, *i.e.*, a topological space X endowed with a continuous map $\delta_X : X\to X\vee X$ which is co-associative (up to homotopy). Note that all suspension spaces has this structure, even though they are rarely manifolds. Then, by functoriality, we get a morphism $\delta_X^* : CH^{X\vee X}(A,B)\to CH^X(A,B)$.

Corollary 5.6. *Assume X is a homotopy coassociative co-H-space. The composition*

$$\cup_X : \quad CH^X(A,B)\otimes CH^X(A,B) \stackrel{\mu_\vee}{\longrightarrow} CH^{X\vee X}(A,B) \stackrel{\delta_X^*}{\longrightarrow} CH^X(A,B),$$

called the cup-product*, induces a structure of graded associative algebra on the cohomology $HH^X(A,B)$. It is further unital if B is unital and X counital.*

Proof. The associativity follows from Proposition 5.5 and Proposition 3.20. When B has an unit 1_B and X is counital, then it follows from the contravariance of Hochschild cochains with respect to maps of pointed spaces that the unit of $\cup_X$ is given by the canonical map

$$k \xrightarrow{1_B} B \cong CH^{pt}(A,B) \xrightarrow{(X\to pt)^*} CH^X(A,B) \tag{16}$$

since, by definition, the two compositions $\Big(id \vee (X \to pt)\Big) \circ \delta_X$ and $\Big((X \to pt) \vee id\Big) \circ \delta_X$ are homotopical to the identity and, further, the composition

$$CH^X(A,B) \otimes k \overset{id\otimes 1_B}{\to} CH^X(A,B) \otimes CH^{pt}(A,B) \overset{\mu_\vee}{\to} CH^X(A,B)$$

is the identity map of $CH^X(A,B)$ (as can be checked on any simplicial set model of X). ☐

In particular, the pinching map $S^d \to S^d \vee S^d$ obtained by collapsing the equator to a point induces a cup product $\cup_{S^d} : CH^{S^d}(A,B) \otimes CH^{S^d}(A,B) \to CH^{S^d}(A,B)$ for Hochschild cohomology over spheres for any E_∞-algebra A and A-algebra B. For CDGA's, this cup-product agrees by definition and Remark 3.22 with the one introduced by the first author in [G].

Example 5.7. In the case of spheres and CDGA's, there is an explicit description of the cup product if one uses the standard model of the dimension d sphere. Recall that the standard simplicial set model of the circle S^1 is the simplicial set, denoted $(S^1_{st})_\bullet$, generated by a unique non-degenerate simplex. Thus $(S^1_{st})_n := n_+$ where $n_+ = \{0, \cdots, n\}$ has $\{0\}$ for its base point see [G, GTZ, P]. The standard simplicial set $(S^d_{st})_\bullet$ is the iterated smash product $(S^d_{st})_\bullet = (S^1_{st})_\bullet \wedge \cdots \wedge (S^1_{st})_\bullet$ so that $(S^d_{st})_n = (n^d)_+$. Using this standard simplicial set model, we have an equivalence

$$CH^{S^d}(A,M) \cong CH^{(S^d_{st})_\bullet}(A,M) \cong Hom_k\big(A^{\otimes (\bullet)^d}, M\big)$$

see [G] (in particular, for the description of the differential on the right hand side). Note that we do not know any simplicial map $(S^d_{st})_\bullet \to (S^d_{st})_\bullet \vee (S^d_{st})_\bullet$ modeling the pinching map. However, there is an obvious map $sd_2((S^d_{st})_\bullet) \to (S^d_{st})_\bullet \vee (S^d_{st})_\bullet$ modeling this map (where $sd_2((S^d_{st})_\bullet)$ is the edgewise subdivision [McC] and can be seen as the simplicial model of the circle obtained by gluing two intervals at their endpoints).

There is however a cochain complex map making $CH^{(S^d_{st})_\bullet}(A,B)$ a differential graded associative algebra on the nose described in [G]. Let $f \in C^{(S^d_{st})_p}(A,B) \cong Hom_k(A^{\otimes \left(p^d\right)}, B)$ and $g \in C^{(S^d_{st})_q}(A,B) \cong Hom_k(A^{\otimes \left(q^d\right)}, B)$. Define $f \cup_0 g \in C^{(S^d_{st})_{p+q}}(A,B) \cong Hom_k\big(A^{\otimes \left((p+q)^d\right)}, B\big)$ by

$$\begin{aligned} f \cup_0 g\Big(&(a_{i_1,\dots,i_d})_{1\le i_1,\dots,i_d \le p+q}\Big) \\ &= f\big((a_{i_1,\dots,i_d})_{1\le i_1,\dots,i_d\le p}\big) g\big((a_{i_1,\dots,i_d})_{p+1\le i_1,\dots,i_d\le p+q}\big) \prod a_{j_1,\dots,j_d} \end{aligned} \tag{17}$$

where the last product is over all indices which are not in the argument of f or g. Note that for $d = 1$, this is the formula of the usual cup-product for Hochschild

cochains as in [Ge] and for $n = 2$, this is the Riemann sphere product as defined in [GTZ].

The following Lemma is proved using a straightforward computation

Lemma 5.8. *Let A be a CDGA and B a commutative differential graded A-algebra. Then $\big(CH^{(S^d_{st})\bullet}(A,B), d, \cup_0\big)$ (where d is the total differential as in*[G, GTZ]*) is an associative differential graded algebra (and unital if B is unital).*

The above explicit formula for the cup-product realizes the cup-product induced by the co-H space structures of the spheres.

Proposition 5.9. *The natural equivalence $CH^{S^d_{st}\bullet}(A,B) \cong CH^{S^d}(A,B)$ is an equivalence of E_1-algebras (with E_1-structures induced by Lemma 5.8 and Corollary 5.6).*

Proof. The proof in the case $d = 2$ is given in the proof of [GTZ, Proposition 3.3.17]. The argument for general d is the same. ☐

We finish this section by giving a more structured version of the wedge product. The wedge product (15) is the degree 0-component of a higher homotopical tower of wedge products. Since B is an E_∞-algebra, it is in particular, by restriction of structure, an E_n-algebra for any positive integer n. Thus, for any $c \in C_*(\mathcal{C}_n(r))$ (where $C_*(\mathcal{C}_n)$ is the little dimension n-cubes operad), we have a map of $A^{\otimes k}$-module $m_B(c) : B^{\otimes r} \longrightarrow B$. Similarly to the wedge product, we can thus define the composition

$$(18) \quad \mu_\vee(c) : \bigotimes_{i=1}^{r} CH^{X_i}(A,B) \longrightarrow Hom_{A^{\otimes r}}\Big(\bigotimes_{i=1}^{r} CH_{X_i}(A), B^{\otimes r}\Big) \stackrel{(m_B(c))_*}{\longrightarrow} Hom_{A^{\otimes r}}\Big(\bigotimes_{i=1}^{r} CH_{X_i}(A), B\Big) \cong CH^{\bigvee_{i=1}^r X_i}(A,B).$$

where the first map is given by the tensor products of functions.

Remark 5.10. When B is a CDGA, then all operations $\mu_\vee(c)$ vanishes if c is not of degree 0.

5.2. **A natural E_d-algebra structure on Hochschild cochains modeled on d-dimensional spheres.** We have already seen the definition of the cup product for Hochschild cochains modeled on spheres for E_∞-algebras, see Corollary 5.6. We now turn to the full E_d-structure on $CH^{S^d}(A,B)$. In [G], the first author proved that if A is a CDGA and B is a commutative A-algebra (for example $B = A$), there is a natural E_n-algebra structure on $CH^{S^n}(A,B)$. In this section we recall this construction in the context of ∞-categories of E_∞-algebras. We will relate this construction to centralizers in the sense of Lurie [L-HA, Lu3] in Section 7.

Recall that we denote $\mathcal{C}_d$ the usual d-dimensional little cubes operad (as an operad of topological spaces) whose associated ∞-operad is a model for $\mathbb{E}_d^{\otimes}$, see[L-HA, Lu3]. $\mathcal{C}_d(r)$ is the configuration space of r many d-dimensional open cubes in I^d. Any element $c \in \mathcal{C}_d(r)$ defines a map $pinch_c : S^d \to \bigvee_{i=1\dots r} S^d$ by collapsing the complement of the interiors of the r cubes to the base point. The maps $pinch_c$ assemble together to give a continuous map

$$(19) \qquad pinch : \mathcal{C}_d(r) \times S^d \longrightarrow \bigvee_{i=1\dots r} S^d.$$

Note that the map *pinch* preserves the base point of S^d hence passes to the pointed category.

For any topological space X, the singular set functor $X \mapsto \Delta_\bullet(X) := Map(\Delta^\bullet, X)$ defines a (fibrant) simplicial set model of X. Hence, applying the singular set functor to the above map *pinch*, the contravariance[10] of Hochschild cochains (see Proposition 3.9 and Proposition 3.20) and the wedge product (18) $\mu_\vee$, we get, for all $r \geq 1$, a morphism

$$(20)\quad \begin{aligned} pinch^*_{S^d,r} : C_*\big(\mathcal{C}_d(r)\big) \otimes \Big(CH^{S^d}(A,B)\Big)^{\otimes r} & \\ \stackrel{diag \otimes id}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big)^{\otimes 2} \otimes \Big(CH^{S^d}(A,B)\Big)^{\otimes k} & \\ \stackrel{\mu_\vee (diag^{(2)})_*}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big) \otimes CH^{\bigvee_{i=1}^r S^d}\big(A,B\big) & \\ \stackrel{pinch^*}{\longrightarrow} CH^{S^d}(A,B) & \end{aligned}$$

in k-Mod_∞. Here $diag : C_*\big(\mathcal{C}_d(r)\big) \to \Big(C_*\big(\mathcal{C}_d(r)\big)\Big)^{\otimes 2}$ is the diagonal and $diag^{(1)}$, $diag^{(2)}$ its components.

Theorem 5.11. *Let A be an E_∞-algebra and B an E_∞-A-algebra (not necesssarily unital). The collection of maps $(pinch_{S^d,k})_{k\geq 1}$ makes $CH^{S^d}(A,B)$ an E_d-algebra (naturally in A, B), which is unital if B is unital. Further, the underlying E_1-structure of $CH^{S^d}(A,B)$ agrees with the one given by Corollary 5.6.*

Proof. To prove the first statement we need to prove that the morphisms $pinch^*_{S^d,r}$ are compatible with the operadic composition in $C_*\big(\mathcal{C}_d(r)\big)$, the singular chains on the little d-dimensional cubes. Since the diagonal $diag : C_*\big(\mathcal{C}_d(r) \to \Big(C_*\big(\mathcal{C}_d(r)\big)\Big)^{\otimes 2}$ is a map of ∞-operads, by Proposition 5.5, this reduces to the commutativity of the following diagram for every $j \in \{1, \dots, k\}$

$$\begin{array}{ccc} \mathcal{C}_d(k) \times \mathcal{C}_d(\ell) \times S^d & \xrightarrow{pinch} & \mathcal{C}_d(\ell) \times \bigvee_{i=1\dots k} S^d \\ \Big\downarrow{\scriptstyle \circ_j \times id_{S^d}} & & \Big\downarrow{\scriptstyle id_{\bigvee_{i=1\dots j-1} S^d} \times pinch \times id_{\bigvee_{i=j+1\dots k} S^d}} \\ \mathcal{C}_d(k+\ell) \times S^d & \xrightarrow{pinch} & \bigvee_{i=1\dots k+\ell} S^d \end{array}$$

in other words it reduces to the fact that the pointed sphere S^d is a $\mathcal{C}_d$-coalgebra in the category of pointed topological spaces endowed with the monoidal structure given by the wedge product.

The underlying E_1-structure is given by any element in $\mathcal{C}_d(2)$ generating the homology group $H_0(\mathcal{C}_d(2), \mathbb{Z}) \cong \mathbb{Z}$. We can, for instance, take the configuration of the two open cubes $(-1,0)^d$ and $(0,1)^d$ in $(-1,1)^d$. It follows immediately with this choice, that the associated E_1-structure is given by the cup-product $\cup_{S^d}$ of Corollary 5.6 up to equivalences of E_1-algebras. The unit is given by the map (16) as in Corollary 5.6.

□

[10]with respect to maps of topological spaces

This theorem will be generalized in Theorem 8.8 below to also include generalized sphere topology operations. The naturality in A and B means that if C is a B-E_∞-algebra map, then, there is an E_d-algebra homomorphism

$$CH^{S^d}(A,B)\otimes CH^{S^d}(B,C)\longrightarrow CH^{S^d}(A,C)$$

see Proposition 7.12 and Theorem 7.7.

Remark 5.12. For $d>1$, Theorem 5.11 implies that the cup-product makes the Hochschild cohomology groups $HH^{S^d}(A,B)$ a graded *commutative* algebra (and not only associative as in the case $d=1$).

Further, when $B=A$ (endowed with its canonical A-algebra structure), the E_d-structure can actually be lifted naturally to an E_{d+1}-structure; see Theorem 7.25.(3).

Remark 5.13. Similarly to Example 5.7, it is possible (but a bit tedious) to give explicit description of the higher $\cup_i$-products on the standard models of the spheres. Details are left to the interested reader.

The core of the proof of Theorem 5.11 is the E_d-co-H-space structure of the sphere. We say that a pointed topological space X is an E_d-co-H-space if it as E_d-coalgebra in the category of pointed spaces with monoidal structure given by the wedge product.

In other words, there are continuous maps $\mathcal{C}_d(k)\times X\to\bigvee_{i=1\dots k}X$ which are compatible with the operadic composition in $\mathcal{C}_d$. Mimicking the proof of Theorem 5.11 gives the following enhancement of Corollary 5.6:

Corollary 5.14. *Let X be an E_d-co-H-space, A an E_∞-algebra and B an E_∞-A-algebra. Then there is a natural (in X in E_d-co-H-space, A and B) E_d-algebra structure on $CH^X(A,B)$ refining the cup-product of Corollary 5.6*

6. Factorization homology and E_n-modules

In this section, for $n=\{1,2,\dots,\}\cup\{+\infty\}$, we collect some results on the category of E_n-modules over an E_n-algebra A. In particular we identify it with left modules over the factorization homology $\int_{S^{n-1}}A$ in § 6.1. Then we apply this to E_∞-modules to show the existence and uniqueness of the lift of Poincaré duality in the category of E_∞-modules in § 6.3. These results are latter used in § 7 and § 8.

6.1. Universal enveloping algebra of an E_n-algebra.

In this section we will recall some general results that are needed, among other places, in the proof of Proposition 7.2. We start with the following very useful result describing the universal enveloping algebra of an E_n-algebra in terms of factorization homology. Note that universal enveloping algebras of E_n-algebras are given by the left adjoint of the forgetful functor $E_n\text{-}Alg\to k\text{-}\mathrm{Mod}_\infty$.

Proposition 6.1 (Francis, Lurie)**.** *Let A be an E_n-algebra ($n\in\mathbb{N}$). The category $A\text{-}Mod^{E_n}$ is equivalent as a symmetric monoidal $(\infty,1)$-category to the category of left modules over the factorization homology $\int_{S^{n-1}}(A)$, with respect to the canonical outward n-framing on $S^{n-1}\subset\mathbb{R}^n$.*

Proof. This is proved in [F1] and can also be found in [Lu3, L-HA]. Note that by the ∞-version of the Barr-Beck theorem [Lu1, L-HA] for any E_n-algebra A, there is an E_n-enveloping algebra $U_A^{(n)}\in E_1\text{-}Alg$ with a natural equivalence $U_A^{(n)}\text{-}LMod\cong$

$A\text{-}Mod^{E_n}$, see *loc. cit* and also [Fre]. Now the result follows from the natural equivalence $U_A^{(n)} \overset{\simeq}{\rightarrow} \int_{S^{n-1}} A$ see [F1, Proposition 3.19]. $\square$

This Lemma extends to the case $n = \infty$, see Lemma 6.8 and more importantly Theorem 6.6 below.

Remark 6.2. In terms of factorization algebra, the equivalence in Proposition 6.1 can be thought of as the pushforward of factorization algebras. The (euclidean) norm of a vector defines a canonical map $N : D^n_* \to [0,1)_*$, where $[0,1)_*$ is the half open interval with a unique closed stratum given by the point 0. The $(\infty,1)$-category of locally constant factorization algebra on the stratified manifold $[0,1)_*$ is equivalent to the $(\infty,1)$-category $LMod$. The above equivalence is induced by the pushforward $N_* : Fac^{lc}_{D^n_*} \to Fac^{lc}_{[0,1)_*}$ by N.

We will later need the following lemma, which expresses the compatibility of the equivalence of categories given by Proposition 6.1 with the inclusions of E_{n+1}-algebras inside E_n-algebras. We feel this lemma is of independent interest anyhow. Suppose X is a codimension 1 submanifold of an n-framed manifold and Y endowed with a trivialization $\psi : X \times \mathbb{R} \hookrightarrow Y$ of a tubular neighborhood in Y. Then, for any E_n-algebra A, there is a canonical map $\psi : \int_X A \to \int_Y A$ (which depends on the trivialization).

Lemma 6.3. *Let A be an E_{n+1}-algebra and $\phi_n : S^{n-1} \times \mathbb{R} \hookrightarrow S^n$ the inclusion of an open (tubular) neighborhood of the equatorial sphere $S^{n-1} = S^n \cap \big(\mathbb{R}^n \times \{0\}\big)$ inside S^n. The following diagram, in which the vertical arrows are given by Proposition 6.1, is commutative,*

$$\begin{tikzcd} A\text{-}Mod^{E_{n+1}} \arrow[r] \arrow[d, "\simeq"] & A\text{-}Mod^{E_n} \arrow[d, "\simeq"] \\ (\int_{S^n} A)\text{-}LMod \arrow[r, "\phi_n^*"] & (\int_{S^{n-1}} A)\text{-}LMod \end{tikzcd}.$$

Proof. The universal property of the E_n-enveloping algebra $U_A^{(n)}$ implies that the map of ∞-operad $\mathbb{E}_n^{\otimes} \to \mathbb{E}_{n+1}^{\otimes}$ (see § 2.2) yields a canonical map of E_1-algebras $U_A^{(n)} \to U_A^{(n+1)}$. It remains to identify the composition $\theta_n : \int_{S^{n-1}} A \cong U_A^{(n)} \to U_A^{(n+1)} \cong \int_{S^n} A$ with ϕ_n to prove the lemma. From the proof of [F1, Proposition 3.19], we know that $U_A^{(n)}$ is computed by the colimit of a (simplicial) diagram,

$$\coprod_{K\in Fin} \mathbb{E}_n^{\otimes}\Big(K \coprod\{pt\}\Big) \otimes A^{\otimes K} \leftleftarrows \coprod_{\mathbb{E}_n^{\otimes}(J,I)} \mathbb{E}_n^{\otimes}\Big(I \coprod\{pt\}\Big) \otimes A^{\otimes J} \cdots . \tag{21}$$

Similarly, $\int_{S^{n-1}} A$ can be computed as the colimit of a similar diagram,

$$\coprod_{K\in Fin} Emb^{fr}\Big(\coprod_K D^n, S^{n-1}\times\mathbb{R}\Big)\otimes A^{\otimes K} \leftleftarrows \coprod_{\mathbb{E}_n^{\otimes}(J,I)} Emb^{fr}\Big(\coprod_I D^n, S^{n-1}\times\mathbb{R}\Big)\otimes A^{\otimes J} \cdots , \tag{22}$$

where Emb^{fr} denotes the space of framed embeddings.

Furthermore, the equivalence $U_A^{(n)} \overset{\simeq}{\rightarrow} \int_{S^{n-1}} A$ is induced by the canonical maps $\mathbb{E}_n^{\otimes}\Big(K \coprod\{pt\}\Big) \to Emb^{fr}\Big(\coprod_K D^n, S^{n-1} \times \mathbb{R}\Big)$ obtained by translating the disk

labeled by the distinguished point to the origin; see the proof of [F1, Proposition 3.19].

The natural map $U_A^{(n)} \to U_A^{(n+1)}$ is induced by the natural maps $\mathbb{E}_n^{\otimes}\Big(K\coprod\{pt\}\Big) \to \mathbb{E}_{n+1}^{\otimes}\Big(K\coprod\{pt\}\Big)$ in diagram (21). Since the natural map of ∞-operads $\mathbb{E}_n^{\otimes} \to \mathbb{E}_{n+1}^{\otimes}$ is given by sending n-dimensional disks D to $D\times\mathbb{R}$, we get commutative diagrams

$$\begin{tikzcd}
\mathbb{E}_n^{\otimes}\Big(K\coprod\{pt\}\Big) \arrow[r] \arrow[d] & \mathbb{E}_{n+1}^{\otimes}\Big(K\coprod\{pt\}\Big) \arrow[d] \\
Emb^{fr}\left(\coprod_K D^n, S^{n-1}\times\mathbb{R}\right) \arrow[r] & Emb^{fr}\left(\coprod_K D^{n+1}, S^{n}\times\mathbb{R}\right)
\end{tikzcd}$$

where the lower map is induced by the embedding

$$\phi_n\times\mathbb{R} : (S^{n-1}\times\mathbb{R})\times\mathbb{R} \hookrightarrow S^n\times\mathbb{R}$$

prescribed in the assumptions of the lemma. It follows that $\theta_n : \int_{S^{n-1}} A \cong U_A^{(n)} \to U_A^{(n+1)} \cong \int_{S^n} A$ is obtained by taking the colimit of these lower maps

$$Emb^{fr}\left(\coprod_K D^n, S^{n-1}\times\mathbb{R}\right) \stackrel{\phi_n\times\mathbb{R}}{\longrightarrow} Emb^{fr}\left(\coprod_K D^{n+1}, S^{n}\times\mathbb{R}\right)$$

applied to diagram (22), which, by definition, is the map $\phi_n : \int_{S^{n-1}} A \to \int_{S^n} A$. $\square$

Remark 6.4. It follows from the axioms of factorization homology (see [Lu3, L-HA, F1] or [GTZ2, Section 6]) that for any E_n-algebra A, there is an natural equivalence $A \cong \int_{D^n} A$ in $k-Mod_\infty$. Since D^n can be expressed as the union[11] of itself with a trivialization of $S^{n-1}\times D^1$, there is a natural left $(\int_{S^{n-1}} A)$-module structure on $\int_{D^n} A$ see [Lu3, L-HA], [F1, Section 3] or [GTZ2, Section 6.3] for details. Note that this left $\big(\int_{S^{n-1}} A\big)$-module structure is given by a map

$$\int_{S^{n-1}} A \otimes \int_{D^n} A \cong \int_{\left(S^{n-1}\times D^1\right)\coprod D^n} A \longrightarrow \int_{D^n} A$$

induced by any embedding $\left(S^{n-1}\times D^1\right)\coprod D^n \hookrightarrow D^n$ mapping D^n onto a subdisk $D(0,r)\subset D^n$ (for some radius $r>0$) and $S^{n-1}\times D^1$ onto a sub-annulus included in $D^n\setminus D(0,r)$.

By Proposition 6.1, we thus get a natural A-E_n-module structure on A which relates to the canonical A-E_n-module structure of A as follows.

Lemma 6.5. *The natural equivalence $A \cong \int_{D^n} A$ is an equivalence of A-E_n-modules.*

Proof. Considered a framed embedding of $D^n \hookrightarrow \mathbb{R}^n$. Since $D^n\setminus\{0\}$ is framed, the result follows from [F1, Remark 3.26]. In fact, the proof of [F1, Proposition 3.19] applied to A and not the unit object of $\mathcal{C} = k\text{-}Mod_\infty$ gives an equivalence of left $\int_{S^{n-1}} A$-modules between A viewed as an $(\int_{S^{n-1}} A)$-module and $\int_{D^n} A$. $\square$

[11] that is, $D^n \cong D^n \cup_{S^{n-1}\times D^1} \emptyset$

6.2. **Application of higher Hochschild chains to prove Theorem 6.6.** For E_∞-algebras, Proposition 6.1 has a simpler and well-known [L-HA, KM, Fre] form, see Theorem 6.6 below. In this section, we recall this result and then give an independent proof using the formalism of factorization homology/higher Hochschild chains.

The following Theorem is due to Lurie [L-HA, Proposition 4.4.1.4], [Lu2] and also appeared independently in the work of Fresse [Fre].

Theorem 6.6. *Let A be an E_∞-algebra. There is an equivalence of symmetric monoidal ∞-categories between the category $A\text{-}Mod^{E_\infty}$ of E_∞ A-Modules and the category of left A-modules (where A is viewed as an E_1-algebra). In particular:*

- *Any left A-module can be promoted into an E_∞-A-module (up to quasi-isomorphisms)*
- *Any map $f : M \to N$ of left A-modules can be lifted to a map of E_∞-modules (up to a contractible family of choices)*

The theorem allows to reduce the study of E_∞-modules on $C^*(X)$ to the study of left modules on the (differential graded) associative algebra $(C^*(X), \cup)$, for instance see § 6.3. Also see Example 6.11 and Remark 6.12 for a more explicit description of the lifts of left modules into E_∞-ones.

Remark 6.7. When A is an E_∞-algebra the categories of left and right modules over A (viewed as an E_1-algebra) are equivalent. Hence one can replace left modules by right modules in Theorem 6.6.

The rest of this section is devoted to an alternative proof of Theorem 6.6 using § 6.1 and higher Hochschild theory. We first start with the following analogue of Proposition 6.1.

Lemma 6.8. *Let A be an E_∞-algebra. The category $A\text{-}Mod^{E_\infty}$ of E_∞-A-Modules is equivalent as a symmetric monoidal $(\infty,1)$-category to the category of left modules over the derived Hochschild chains $CH_{S^\infty}(A)$, viewed as an E_1-algebra by forgetting extra structure.*

Proof. By Theorem 3.11, there is a canonical equivalence $\int_{S^n} A \cong CH_{S^n}(A)$ for any $n \in \mathbb{N}$.

The maps of operad $\mathbb{E}_i^\otimes \to \mathbb{E}_{i+1}^\otimes$ are induced by the maps $\mathbb{R}^i \cong \mathbb{R}^i \times \{0\} \hookrightarrow \mathbb{R}^{i+1}$ which, by restriction induces canonical maps $S^{i-1} \cong S^i \cap \big(\mathbb{R}^i \times \{0\}\big) \hookrightarrow S^i$, and, by functoriality, maps $\phi_i : CH_{S^{i-1}}(A) \to CH_{S^i}(A)$.

By Lemma 6.3 (and Theorem 3.11), we get a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r, hook] & A\text{-}Mod^{E_{n+1}} \arrow[r, hook] \arrow[d, "\simeq"] & A\text{-}Mod^{E_n} \arrow[r, hook] \arrow[d, "\simeq"] & \dots \arrow[r, hook] & A\text{-}Mod^{E_1} \arrow[d, "\simeq"] \\
\dots \arrow[r] & CH_{S^n}(A)\text{-}LMod \arrow[r, "\phi_n^*"] & CH_{S^{n-1}}(A)\text{-}LMod \arrow[r] & \dots \arrow[r, "\phi_0^*"] & CH_{S^0}(A)\text{-}LMod
\end{tikzcd}.$$

From Lemma 6.9, we deduce a natural equivalence

$$A\text{-}Mod^{E_\infty} \;\cong\; \lim_{n\geq 1} CH_{S^n}(A)\text{-}LMod.$$

Mimicking the proof of [GTZ2, Lemma 5.1.3], we get a natural equivalence

$$\varinjlim \Big(CH_{S^0}(A) \to CH_{S^1}(A) \to \cdots \to CH_{S^n}(A) \to \dots \Big) \xrightarrow{\simeq} CH_{S^\infty}(A).$$

It follows that we have an equivalence $CH_{S^\infty}(A)\text{-}LMod \to \lim_{n\geq 1} CH_{S^n}(A)\text{-}LMod$ and the lemma follows. $\square$

Lemma 6.9. *Let A be an E_∞-algebra, then $CH_{S^\infty}(A)$ is canonically equivalent to A as an E_∞-algebra. In particular, there is a canonical equivalence*

$$A\text{-}Mod^{E_n} \;\cong\; CH_{S^\infty}(A) - Mod^{E_n}$$

for any $n \in \{0, 1, \dots, \infty\}$.

Proof. It follows from Theorem 3.26 since S^∞ is a deformation retract of a point. $\square$

The canonical map $\mathbb{E}_{n-1}^{\otimes} \to \mathbb{E}_n^{\otimes}$ yields a natural functor $A\text{-}Mod^{E_n} \to A\text{-}Mod^{E_{n-1}}$ for any E_n-algebra A.

Lemma 6.10. *Let A be an E_∞-algebra. Then $A\text{-}Mod^{E_\infty}$ is the (homotopy) limit*

$$A\text{-}Mod^{E_\infty} \cong lim\Big(\cdots \to A\text{-}Mod^{E_n} \to A\text{-}Mod^{E_{n-1}} \to \cdots \to A\text{-}Mod^{E_1}\Big).$$

Proof. Recall that $\mathbb{E}_\infty^{\otimes} \cong \underset{n\geq 1}{colim}\, \mathbb{E}_n^{\otimes}$ [Lu3, L-HA]. Since we have commuting restriction maps $A\text{-}Mod^{E_\infty} \to A\text{-}Mod^{E_n}$ $(n \in \mathbb{N})$, there is a canonical map

$$\tau : A\text{-}Mod^{E_\infty} \longrightarrow \underset{n\geq 1}{lim}\, A\text{-}Mod^{E_n}.$$

We want to prove that this map τ is an equivalence. Given any E_n-algebra A and an E_n-A-module M, the trivial extension $A \oplus M$ has a natural structure of E_n-algebra. The trivial extension functor $M \mapsto A \oplus M$ is a (natural in A) equivalence of ∞-categories between $A\text{-}Mod^{E_n}$ and $E_n\text{-}Alg_{/A}$ which, by naturality, commutes with the restriction of structure functors $A\text{-}Mod^{E_n} \to A\text{-}Mod^{E_{n-1}}$ and $E_n\text{-}Alg_{/A} \to E_{n-1}\text{-}Alg_{/A}$. It follows that any object of $\lim_{n\geq 1} A\text{-}Mod^{E_n}$ is equivalent to an object of $\underset{n\geq 1}{lim}\, E_n\text{-}Alg_{/A}$. Such an object is a (homotopy type of) chain complex equipped with compatible E_n-structures for all $n \geq 1$, thus is an E_∞-algebra. It is also endowed with compatible augmentations of E_n-algebras to A. Hence we get a map

$$\varphi : \underset{n\geq 1}{lim}\, E_n\text{-}Alg_{/A} \longrightarrow E_\infty\text{-}Alg_{/A}$$

which is a quasi-inverse of the canonical map

$$\tau : E_\infty\text{-}Alg_{/A} \longrightarrow \underset{n\geq 1}{lim}\, E_n\text{-}Alg_{/A}$$

induced by the restrictions functors. The result now follows by applying the (quasi-inverse of the) trivial extension functor. $\square$

Proof of Theorem 6.6. The first statement follows from Lemmas 6.8 and 6.9 and the last two statements are consequences of the first one. $\square$

Example 6.11. Let A be an E_∞-algebra and M be a left $CH_{S^\infty}(A)$-module (here $CH_{S^\infty}(A)$ is equipped with its canonical E_1-structure by restriction of structure). Since $CH_{S^\infty}(A)$ is in fact an E_∞-algebra, for any $n \in \{1, \dots, \infty\}$, it is canonically equivalent to its opposite E_n-algebra $CH_{S^\infty}(A)^{op}$. The equivalence is explicitly given by the antipodal map $S^\infty \overset{\text{ant}}{\to} S^\infty$ (and functoriality of Hochschild

chains). Thus, there is a canonical structure of $CH_{S^\infty}(A)\otimes CH_{S^\infty}(A)$-$E_\infty$-modules on $CH_{S^\infty}(A)$. By restriction of structure, the map

$$CH_{S^\infty}(A) \stackrel{1\otimes id}{\longrightarrow} CH_{S^\infty}(A)\otimes CH_{S^\infty}(A)$$

endows $CH_{S^\infty}(A)$ with a right module structure over itself (viewed as an E_1-algebra) which commutes with the $CH_{S^\infty}(A)$-module structure induced by the map

$$CH_{S^\infty}(A) \stackrel{id\otimes 1}{\longrightarrow} CH_{S^\infty}(A)\otimes CH_{S^\infty}(A).$$

This extra structure of $CH_{S^\infty}(A)$ endows the tensor product (of a right and left module over $CH_{S^\infty}(A)$ viewed as an E_1-algebra)

$$CH_{S^\infty}(A)\otimes_{CH_{S^\infty}(A)} M$$

with a structure of E_∞-module.

Remark 6.12. One can lift any left A-module to an A-E_∞-module in the same way as in Example 6.11.

By restriction of structure, any left A-module map between E_∞-A-modules can be lifted to a map of E_n-modules (for $n\in\mathbb{N}\cup\{\infty\}$). For the sake of explicit computations we now explain how to realize this concretely using the higher Hochschild functor. Let M, N be E_∞-modules over A. By restriction of structure we get in particular left A-modules structure on M and N. Let $f: M\to N$ be a map of left A-modules. The natural structure of $A\otimes A^{op}$-E_∞-module structure on A yields, by restriction of structure, Proposition 6.1 and Lemma 3.11, a natural structure of left $A\otimes\Big(CH_{S^{n-1}}(A)\Big)^{op}$-module on A where the left factor A is viewed as an E_1-algebra only. It follows that, viewing N as left A-module only by restriction, $Hom_A(A,N)$ is endowed with a natural left $CH_{S^{n-1}}(A)$-module structure and further that we have a natural isomorphism of left $CH_{S^{n-1}}(A)$-modules $Hom_A(A,N)\stackrel{\simeq}{\to} N$ (given by $f\mapsto f(1)$). We get similarly a left $CH_{S^{n-1}}(A)\otimes A^{op}$-module structure on A and a natural equivalence of left $CH_{S^{n-1}}(A)$-modules $A\otimes_A M\stackrel{\simeq}{\to} M$ (where the tensor product is over A viewed as an E_1-algebra only).

We now explain how to lift f to an E_n-module map (here $n\in\{1,\dots,\infty\}$). The canonical map $D^n\to pt$ being a homotopy equivalence, we get a natural quasi-isomorphism $CH_{D^n}(A)\stackrel{\sim}{\to} A$ with quasi-inverse induced by the map sending a point to the center of D^n. The canonical map $S^{n-1}\hookrightarrow D^n$ given by the boundary of D^n gives a map of E_∞-algebra $CH_{S^{n-1}}(A)\to CH_{D^n}(A)$ which, together with the previous morphism, endow $CH_{D^n}(A)$ with a structure of left $CH_{S^{n-1}}(A)\otimes A^{op}$-module. We thus have a natural quasi-isomorphism (of chain complexes)

$$\begin{aligned} Hom_{CH_{S^{n-1}}(A)}(M,N) &\cong Hom_{CH_{S^{n-1}}(A)}\big(A\otimes_A M, Hom_A(A,N)\big)\\ &\stackrel{\sim}{\longrightarrow} Hom_{CH_{S^{n-1}}(A)}\big(CH_{D^n}(A)\otimes_A M, Hom_A(A,N)\big)\\ &\stackrel{\simeq}{\longrightarrow} Hom_A\big(A\otimes_{CH_{S^{n-1}}(A)} CH_{D^n}(A)\otimes_A M, N)\big)\end{aligned}$$

where the last map is the canonical isomorphism

$$\psi\mapsto\Big(x\otimes_{CH_{S^{n-1}}(A)} y\otimes_A m\mapsto \pm\psi(y\otimes_A m)(x)\Big)$$

where the sign $\pm$ is given by the Koszul-Quillen signs rule. Note that there is an equivalence of E_∞-algebras $A\otimes_{CH_{S^{n-1}}(A)} CH_{D^n}(A)\stackrel{\simeq}{\leftarrow} CH_{S^n}A$ which induces, by

restriction, a quasi-isomorphism of left $A\otimes A^{op}$-modules (induced by the choice of two antipodal points on S^n). We thus get a quasi-isomorphism

$$Hom_A\big(A\otimes_{CH_{S^{n-1}}(A)} CH_{D^n}(A)\otimes_A M,N)\big)\stackrel{\simeq}{\longrightarrow} Hom_A\big(CH_{S^n}(A)\otimes_A M,N)\big)$$

hence an explicit quasi-isomorphism

$$Hom_{CH_{S^{n-1}}(A)}(M,N)\stackrel{\simeq}{\longrightarrow} Hom_A\big(CH_{S^n}(A)\otimes_A M,N)\big). \tag{23}$$

The canonical map $S^n\to pt$ also yield a map of E_∞-algebras $CH_{S^n}(A)\to A$, which, by restriction of structures is also a map of left $A\otimes A^{op}$-modules. Hence; we have a natural morphism

$$Hom_A(M,N)\cong Hom_A(A\otimes_A M,N)\longrightarrow Hom_A\big(CH_{S^n}(A)\otimes_A M,N)\big). \tag{24}$$

Thus, for any n, we can lift the left module map $f\in Hom_A(M,N)$ to a map of left $CH_{S^{n-1}}(A)$-module hence a map of A-E_n-module (by Proposition 6.1 or Lemma 6.8). Note that by Lemma 6.9 the map $CH_{S^\infty}(A)\to A$ is a quasi-isomorphism, hence the map (24) is a quasi-isomorphism for $n=\infty$ and the lift of f is unique in that case. However, lift of f to E_n-module maps are not unique in general for finite n.

Remark 6.13. When A is a CDGA, the Hochschild chain complex $CH_{D^n}(A)$ is a semi-free module over $CH_{S^{n-1}}(A)$ (provided we choose a simplicial model $D^n_\bullet$ for D^n and take $\partial D^n_\bullet$ as a model for S^{n-1}), and therefore all equivalences involved in the maps (23) and (24) can be (quasi-)inverted by standard homological algebra techniques. Note that when $A=C^*(X)$ is the algebra of cochains for a topological space X, the map of E_∞-algebras $CH_{S^n}(A)\to A$ can be factorized as a map

$$CH_{S^n}(C^*(X))\to C^*(Map(S^n,X))\to C^*(X)$$

where the last map is induced by the map $X\to Map(S^n,X)$ that sends every point in $p\in X$ to a constant map $C_p: S^n\to X$ defined as $C_p(a)=p$. Hence, in the special case $n=1$, we recover the construction of [FTV], which was done for $M=C^*(X)$ and $N=C_*(X)$ only.

6.3. **Poincaré Duality as a map of E_∞-modules.** We now apply the results of the previous sections to achieve an E_∞-lift of the Poincaré duality isomorphism for a closed manifold.

Let C be an E_∞-coalgebra and let $C^\vee=Hom_k(C,k)$ be its linear dual endowed with its canonical E_∞-algebra structure; in particular, $C^\vee$ is naturally a E_∞-$C^\vee$-module. Similarly, the dual space $(C^\vee)^\vee$ is E_∞-$C^\vee$-module. Note that $C\subset (C^\vee)^\vee$ has an induced E_∞-$C^\vee$-module structure. If C is an E_1-coalgebra, then $C^\vee$ is an E_1-algebra has well.

We recall the following standard definition of the cap-product

Definition 6.14. Let C be an E_1-coalgebra. The *cap-product* is the composition

$$\cap: C^\vee\otimes C\stackrel{id\otimes\Delta}{\longrightarrow} C^\vee\otimes C\otimes C\stackrel{\langle -,-\rangle\otimes id}{\longrightarrow} C$$

where $\Delta: C\to C\otimes C$ is the coproduct (given by the E_1-structure of C) and $\langle -,-\rangle: C^\vee\otimes C\to k$ is the duality pairing. The cap-product of $x\in C^\vee$, $y\in C$ will be denoted $x\cap y$ as usual.

The cap-product map $\cap : C^\vee \otimes C \to C$ allows to associate to any cycle c in C, a map of *left $C^\vee$-modules* $\cap c : C^\vee \to C$, $x \mapsto x \cap c$, called *the cap-product by c*. Note that this construction only uses the underlying E_1-coalgebra structure of C (even if C is an E_∞-algebra).

Corollary 6.15. *Let C be an E_∞-coalgebra. The cap product by c, $C^\vee \xrightarrow{\cap c} C$, lifts uniquely to a map of E_∞-modules $\rho_c : C^\vee \to C$ which is an equivalence if $\cap c$ is a quasi-isomorphism.*

Proof. The cap-product by c, denoted $\cap c : C^\vee \to C$, is a map of left modules over $C^\vee$ (seen as an E_1-algebra) because $\Delta : C \to C \otimes C$ is an E_1-coalgebra structure. It follows from Theorem 6.6 that the unique lift exists. If $\cap c$ is a quasi-isomorphism, then it is an invertible element in $Hom_{C^\vee}(C^\vee, C)$ and thus its lift is invertible in $Hom_{CH_{S^\infty}(C^\vee)}(C^\vee, C)$ (see Remark 6.12 for an explicit description of the equivalence). $\square$

We now specialize to the case where C is the singular cochain of a space. Let us recall the following definition.

Definition 6.16. By a *Poincaré duality space*, we mean a topological space X together with a choice of cycle $[X] \in C_d(X)$ (for some integer d) such that that cap-product $C^*(X) \xrightarrow{\cap[X]} C_{d-*}(X)$ by $[X]$ is a quasi-isomorphism. The integer d is called the *dimension* of X and denoted $d = \dim(X)$.

Example 6.17. An oriented[12] closed manifold M of dimension $\dim(M)$ (in the usual manifold sense of dimension) is a Poincaré duality space of dimension $\dim(M)$.

Remark 6.18. By definition, the cap product by a class $[X]$ is given by $f \mapsto \sum f\big([X]^{(1)}\big)\,[X]^{(2)}$ (where we denote $\Delta([X]) := \sum [X]^{(1)} \otimes [X]^{(2)}$ the coproduct). It follows that the image $\chi_X\big(H^*(X)\big)$ is a finitely generated sub k-module of $H_*(X)$. Thus, if X is a Poincaré duality space, its (co)homology groups are finitely generated (as k-modules).

Let X be a Poincaré duality space (for instance, an oriented closed manifold) with fundamental class $[X]$. Recall that $C_*(X)$ is the singular cochains of X with its natural structure of E_∞-coalgebra (Example 2.5). Its linear dual $C^*(X)$ is endowed with the dual E_∞-algebra structure. Then, by Corollary 6.15 we have

Corollary 6.19. *Let $(X, [X])$ be a Poincaré duality space. The cap-product by $[X]$ induces a quasi-isomorphism of E_∞-$C^*(X)$-modules*

$$\chi_X : C^*(X) \xrightarrow{\simeq} C_*(X)[\dim(X)] \tag{25}$$

realizing the (unique) E_∞-lift of the Poincaré duality isomorphism.

In other words, a Poincaré duality space X (in the sense of Definition 6.16) gives rise to a canonical equivalence of E_∞-modules between its singular chains and cochains.

Definition 6.20. Let $(X, [X])$, $(Y, [Y])$ be Poincaré duality space (of same dimension $d = \dim(X) = \dim(Y)$). A map of Poincaré duality space $f : (X, [X]) \to$

[12] with respect to the homology with coefficient in the ground ring k

$(Y, [Y])$ is a map of topological spaces $f : X \to Y$ such that the following diagram is commutative

$$\begin{tikzcd} C^*(X) \arrow[r, "\cap[X]"] & C_*(X)[d] \arrow[d, "f_*"] \\ C^*(Y) \arrow[u, "f^*"] \arrow[r, "\cap[Y]"] & C_*(Y)[d] \end{tikzcd}$$

in $C^*(Y) - Mod^{E_\infty}$.

Example 6.21. Let $f : M \to N$ be a continuous map between oriented smooth manifolds such that $f_*([M]) = [N]$. Then f induces a map of Poincaré duality spaces.

7. E_n-Hochschild cohomology and centralizers

Given any map $f : A \to B$ of E_∞-algebras, by Theorem 5.11, there is a natural E_n-algebra structure on $CH^{S^n}(A, B)$. On the other hand, for a map $f : A \to B$ of E_n-algebras, Lurie [L-HA, Lu3] constructs an E_n-algebra $\mathfrak{z}(f)$. We prove in § 7.3 that $CH^{S^n}(A, B)$ is equivalent to $\mathfrak{z}(f)$ as an E_n-algebra. This will be a corollary of a more general construction for E_n-Hochschild cohomology. Indeed, Hochschild cochains modeled on spheres $CHS^n(A, B)$ is a special case of E_n-Hochschild cohomology $HH_{\mathcal{E}_n}(A, B)$ of A, B viewed as E_n-algebras, see § 7.1. In Section 7.2, in the general case of a map $f : A \to B$ between E_n-algebras, we will give an explicit E_n-algebra structure on $HH_{\mathcal{E}_n}(A, B)$, similar to the one obtained in Section 5.2. We then prove that $HH_{\mathcal{E}_n}(A, B)$ is equivalent to $\mathfrak{z}(f)$. We will apply these results to the case $A = B$, *i.e.*, to get solutions of (higher) Deligne conjecture in § 7.4.

7.1. E_n-Hochschild cohomology and Hochschild cohomology over S^n.

There is an (operadic) notion of cohomology for E_n-algebras closely related to their deformation complexes, see [F1, KS]. We start with the following definition.

Definition 7.1. Let M be an E_n-A-module over an E_n-algebra A. The E_n-Hochschild complex of A with values M, denoted by $HH_{\mathcal{E}_n}(A, M)$, is by definition (see [F1]) $RHom^{\mathcal{E}_n}_A(A, M)$. Here $RHom^{\mathcal{E}_n}_A$ denotes the hom space in the (∞-)category A-Mod^{E_n} of E_n-A-modules.

In particular, if A is an E_m-algebra with $m \in \{n, n+1 \dots, \infty\}$ (for instance a CDGA), we can define the E_n-Hochschild complex of A $HH_{\mathcal{E}_n}(A, A)$.

In the case where A is an E_∞-algebra, its E_n-Hochschild complex can be described by higher Hochschild cochains over the n-dimensional sphere S^n:

Proposition 7.2. *If A is an E_∞-algebra and M an E_∞-A-module, there is a natural equivalence*

$$HH_{\mathcal{E}_n}(A, M) \cong CH^{S^n}(A, M),$$

where CH^{S^n} denotes the derived higher Hochschild cochain functor.

Proof. Given left modules M, N over an E_1-algebra R, we write $RHom^{left}_R(M, N)$ for the hom space in the (∞,1)-category R-$LMod$ of left R-modules. By Proposition 6.1, there is an equivalence of ∞-categories A-$Mod^{E_n} \cong \left(\int_{S^{n-1}} A\right)$-$LMod$ where $\int_{S^{n-1}} A$ is the factorization homology of S^{n-1} with value in A. Here S^{n-1} is endowed with the n-framing induced by the natural embedding $S^{n-1} \hookrightarrow \mathbb{R}^n$. Thus we have a sequence of natural equivalences

$$\begin{aligned}
HH_{\mathcal{E}_n}(A,M) &\cong RHom_A^{\mathcal{E}_n}(A,M)\\
&\cong RHom^{left}_{\int_{S^{n-1}}A}(A,M)\\
&\cong RHom^{left}_{\int_{S^{n-1}}A}\left(\int_{D^n}A, M\right)\\
&\cong RHom^{left}_{CH_{S^{n-1}}(A)}(CH_{D^n}(A),M)\\
&\cong RHom^{left}_A\left(CH_{D^n}(A)\otimes^{\mathbb{L}}_{CH_{S^{n-1}}(A)}A, M\right)\\
&\cong RHom^{left}_A(CH_{S^n}(A),M)\\
&\cong CH^{S^n}(A,M).
\end{aligned}$$

Here we are using the natural equivalence of E_n-A-modules $\int_{D^n}A \stackrel{\simeq}{\to} A$ (Lemma 6.5).

Note that, by Theorem 3.11, when A is further an E_∞-algebra, we get a natural equivalence of E_1-algebras $\int_{S^{n-1}}(A)\cong CH_{S^{n-1}}(A)$ and by Theorem 3.26 a natural equivalence of E_∞-algebras $CH_{S^n}(A)\cong CH_{D^n}(A)\otimes^{\mathbb{L}}_{CH_{S^{n-1}}(A)}A$. □

Remark 7.3. Let A,B be E_n-algebras and $f:A\to B$ an E_n-algebra map so that B inherits an A-E_n-module structure. By Definition 7.1, Proposition 6.1 and Lemma 6.5, we have natural equivalences

$$HH_{\mathcal{E}_n}(A,B)\ \cong\ RHom_A^{\mathcal{E}_n}\Big(A,B\Big)\ \cong\ RHom^{left}_{\int_{S^{n-1}}A}\Big(\int_{D^n}A,\int_{D^n}B\Big).$$

7.2. **The E_n-algebra structure on $\mathcal{E}_n$-Hochschild cohomology $HH_{\mathcal{E}_n}$(A,B).** In this section, we construct an explicit E_n-algebra structure on the $\mathcal{E}_n$-Hochschild cohomology $HH_{\mathcal{E}_n}(A,B)$ of an E_n-algebra A with value in an E_n-algebra B endowed with an A-E_n-module structure given by a map $A\to B$ of E_n-algebras.

We fix a map $f:A\to B$ of E_n-algebras and we endow B with the induced A-E_n-module structure so that we have E_n-Hochschild cohomology[13] $HH_{\mathcal{E}_n}(A,B)$.

Recall from Section 2.4 (and [Lu3, L-HA]) that giving an E_n-algebra structure to $HH_{\mathcal{E}_n}(A,B)\cong RHom_A^{\mathcal{E}_n}\Big(A,B\Big)$ is equivalent to giving a structure of locally constant factorization algebra on D^n whose global section[14] are $RHom_A^{\mathcal{E}_n}\Big(A,B\Big)$. That is, we need to associate to any disk $U\subset D^n$ a chain complex $HH_{\mathcal{E}_n}\Big(A,B\Big)(U)$ naturally quasi-isomorphic to $RHom_A^{\mathcal{E}_n}\Big(A,B\Big)$ equipped with natural chain maps from

$$\rho_{U_1,\dots,U_\ell,V}\, HH_{\mathcal{E}_n}\Big(A,B\Big)(U_1)\otimes\cdots\otimes HH_{\mathcal{E}_n}\Big(A,B\Big)(U_\ell)\to HH_{\mathcal{E}_n}\Big(A,B\Big)(V) \tag{26}$$

for any pairwise disjoint (embedded) sub-disks U_i in a bigger disk V.

Let $\mathcal{A}$, $\mathcal{B}$ be the underlying locally constant factorization algebras on D^n associated to A and B given by Proposition 2.15 and still denote $f:\mathcal{A}\to\mathcal{B}$ the induced map of factorization algebras. In other words, we assume A, B and f are given by locally constant factorization algebras as in Section 2.4. Similarly, given any map of A-E_n-modules $g:A\to B$, we have (by Proposition 2.20), the associated map

[13] which depends on the map $f:A\to B$ even though it is not explicitly written in the notation

[14] ,*i.e.*, its factorization homology over the whole disk D^n

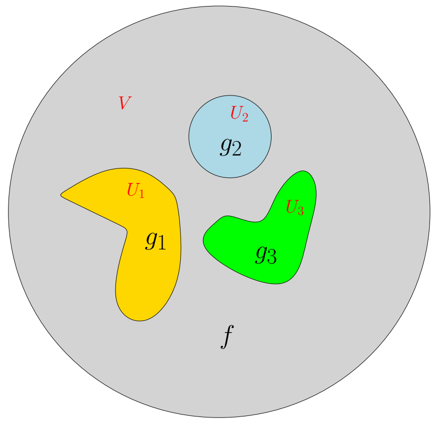

FIGURE 1. The factorization algebra map $\mathcal{A}_{|V} \to \mathcal{B}_{|V}$ obtained by applying the relevant maps of modules g_1, g_2, g_3 (viewed as maps of factorizations algebras) and the E_n-algebra map $f : A \to B$ on the respective regions

$g : \mathcal{A} \to \mathcal{B}$ of (stratified) factorization algebras, as well as, by Proposition 6.1S a map of (left) $\int_{S^{n-1}} A$-modules $g : \int_{D^n} A \to \int_{D^n} B$.

Remark 7.4 (*Sketch of the construction*)**.** We first sketch the idea of the construction. For any sub-disk U_i, we can think of $HH_{\mathcal{E}_n}(A,B) \cong RHom_A^{\mathcal{E}_n}(A,B)$ as the space of stratified factorization algebras maps on the disk U_i (with a distinguished point $*_i$, see Proposition 2.20). Hence, given $g_1, \dots, g_\ell \in HH_{\mathcal{E}_n}(A,B)$, we define the structure map (26) $\rho_{U_1,\dots,U_\ell,V}(g_1,\dots,g_\ell)$ to be the factorization algebra map which, to any sub-disk D inside a given U_i associates $g_i(D)$ and, to any disk D inside (a small neighborhood of) the complement of the U_i's associates $f(D)$. The family of those disks is a basis of all disks inside V, so that such a rule does define a factorization algebra map, which underlies a map of A-E-modules (see Remark 2.21). This is roughly described in Figure 1.

We now define the locally constant factorization algebra on $\mathbb{R}^n$. For any open subset U, the restrictions $\mathcal{A}_{|U}$, $\mathcal{B}_{|U}$ are locally constant factorization algebras[15] on U, and $f_{|U} : \mathcal{A}_{|U} \to \mathcal{B}_{|U}$ a factorization algebra morphism. Thus, if U is a disk[16], $\mathcal{A}(U)$ ($\cong \int_U A$) is an E_n-algebra and $f_{|U} = \int_U f$ makes $\mathcal{B}(U) \cong \int_U B$ an $\mathcal{A}(U)$-E_n-module[17].

Thus to any open disk U, we can associate the following object of k-Mod$_\infty$:

$$RHom_A^{\mathcal{E}_n}\Big(A,B\Big)(U) := RHom_{\mathcal{A}(U)}^{\mathcal{E}_n}\Big(\int_U A, \int_U B\Big). \tag{27}$$

Note that $RHom^{\mathcal{E}_n}\Big(A,B\Big)(U)$ is pointed since our starting map of E_n-algebras $f : A \to B$ induces a canonical element $\int_U f \in RHom_A^{\mathcal{E}_n}\Big(A,B\Big)(U)$. For pairwise

[15] quasi-isomorphic to A and B by definition

[16] that is an open set homeomorphic to a disk

[17] in this section we will write $\int_U A$ for $\mathcal{A}(U)$ viewed as an E_n-module over itself and reserve the notation $\mathcal{A}(U)$ when we think of it as an E_n-algebra

disjoints disks $U_1, \dots, U_r$ included in a larger disk D, we define the structure map

(28)
$$RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U_1)\otimes\cdots\otimes RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U_r) \stackrel{\rho_{U_1,\dots,U_r,D}}{\longrightarrow} RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(D)$$

as follows. We use $U_1, \dots, U_n$ to see the factorization algebra $\mathcal{A}$ restricted to D, denoted $\mathcal{A}_{|D}$, as obtained by gluing together $r+1$-factorization algebras on D (see § 2.3 and [CG]). Note that the $\mathcal{A}(U)$-E_n-module structure on $\int_U A$ allows to see $\mathcal{A}_{|U}$ as a stratified factorization algebra on U with a closed strata given by a point $*_i$ (or a sub-disk); different choices of points leads to canonically equivalent $\mathcal{A}_{|U}$-modules structures. We can choose a collar c_i in the neighborhood of the boundary (in D) of each U_i such that $c_i \cong S^{n-1} \times (0, \epsilon)$ for a homeomorphism induced by a homeomorphism $U_i \cong D^n$; for instance we just choose c_i to be the complement of the point $*_i \in U_i$ (or a suitable sub-disk). This way we get a connected open set

$$U_\partial := D \setminus \big(\coprod_{i=1}^{r}(U_i - col_i)\big)$$

(the notation ∂ is meant to suggest that U_∂ is the boundary of $\coprod U_i$ in D; below we will sometimes refer to it using this terminology). By definition, the U_i's and U_∂ cover D, hence the factorization algebra $\mathcal{A}_{|D}$ is obtained as the gluing of the restricted factorizations algebras $\mathcal{A}_{|U_1}, \dots, \mathcal{A}_{|U_r}$ and $\mathcal{A}_{|U_\partial}$.

Let be given maps $g_i : \int_{|D_i} A \to \int_{|D_i} B$ of (left) $\mathcal{A}(U_i)$-modules ($i = 1 \dots r$) and also denotes $g_i : \mathcal{A}_{|D_i} \to \mathcal{B}_{|D_i}$ the induced maps of (stratified) factorization algebras. We also denote $f_\partial : \mathcal{A}_{|U_\partial} \to \mathcal{B}_{|U_\partial}$ the restriction of $f : \mathcal{A} \to \mathcal{B}$ to U_∂.

Lemma 7.5. *The family $(g_1, \dots, g_r, f_\partial)$ of maps of factorization algebras glues together to define a map*

$$\rho_{U_1,\dots,U_r,D}\,(g_1, \dots, g_r) \in RHom(\mathcal{A}_{|D}, \mathcal{B}_{|D})$$

which is independent in k-Mod_∞ of the choices (of collars) involved. Further, on global sections, the induced map $\int_D \rho_{U_1,\dots,U_r,D}\,(g_1, \dots, g_r) : \int_D A \to \int_D B$ is a map of $\mathcal{A}(D)$-E_n-modules.

It follows (from the above Lemma 7.5) that we get a well-defined structure map $(g_1, \dots, g_r) \mapsto \rho_{U_1,\dots,U_r,D}\,(g_1, \dots, g_r)(D)$, simply denoted by

(29)
$$\rho_{U_1,\dots,U_r,D} : RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U_1) \otimes \cdots \otimes RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U_r) \longrightarrow RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(D)$$

Proof of Lemma 7.5. We first check that the maps $(g_1, \dots, g_r, f_\partial)$ do glue. Note that all triples intersections in the family $(U_1, \dots, U_r, U_\partial)$ are empty and that the only non-empty intersections are those of the form $U_i \cap U_\partial = col_i \cong S^{n-1} \times (0, \epsilon)$. Hence, by definition of the gluing of factorization algebras, we only have to check that the maps g_i and f_∂ are equivalent on $\mathcal{A}_{|(U_i \cap U_\partial)}$. By assumption, the map $g_i : \int_{U_i} A \to \int_{U_i} B$ is a map of $\mathcal{A}(U_i)$-modules (applying Proposition 6.1 and Lemma 6.5, this is equivalent to saying that $g_i : \mathcal{A}(U_i) \to \mathcal{B}(U_i)$ is an $\mathcal{A}(U_i)$-E_n-module map).

It follows from Proposition 6.1 (and Theorem 2.14) that the map of factorization algebras $g_i : \mathcal{A}_{|col_i} \to \mathcal{B}_{|col_i}$ is equivalent to the map induced by the $\mathcal{A}(U_i)$-module structure of $\int_{U_i} B \cong B$. Since this module structure is given by $f : \mathcal{A} \to \mathcal{B}$, it

follows that $\big(g_i\big)_{|col_i}$ is equivalent to $\big(f_\partial\big)_{|col_i}$. Hence the collection $(g_1,\dots,g_r,f_\partial)$ assembles to give an object in $RHom(\mathcal{A}_{|D},\mathcal{B}_{|D})$. Further, we also just proved, that for any choice of collar col'_i in the disk U_i, the value of g_i on $\mathcal{A}_{col'_i}$ is given by f. It is thus independent of the choice of the collar.

Choosing a closed sub-disk $\tilde{D}$ containing $U_1\coprod\cdots\coprod U_r$, we get an open subset $D\setminus\tilde{D}\cong S^{n-1}\times(0,\epsilon')$ in the complement of the U_i's. To check that the induced map $\rho_{U_1,\dots,U_r,D}\,(g_1,\dots,g_r)\big(D\big):A\to B$ is a map of $\mathcal{A}(D)$-E_n-module, by Proposition 6.1, we are left to check that the induced morphism $\int_D A\to\int_D B$ is a morphism of left $\int_{S^{n-1}}A$-modules. Since $\int_{S^{n-1}}A$ is the section $\mathcal{A}\big(S^{n-1}\times(0,\epsilon')\big)$ (Theorem 2.14), we are left to prove that the map $\rho_{U_1,\dots,U_r,D}\,(g_1,\dots,g_r)$ restricted to $D\setminus\tilde{D}\cong S^{n-1}\times(0,\epsilon')$ is equivalent to f, which is obvious since $D\setminus\tilde{D}\subset U_\partial$ and $\big(\rho_{U_1,\dots,U_r,D}\,(g_1,\dots,g_r)\big)_{|U_c}=f_\partial$ by construction. ☐

Remark 7.6. Let us consider the case of the inclusion of the empty set $\emptyset$ inside a disk D. Unwinding the definition of the structure map

$$\rho_{\emptyset,D}:k\cong RHom_A^{\mathcal{E}_n}\Big(A,B\Big)(\emptyset)\longrightarrow RHom_A^{\mathcal{E}_n}\Big(A,B\Big)(D)$$

we see immediately that $\rho_{\emptyset,D}(1)=\int_D f$, in other words 1 is mapped to the base point of $RHom^{\mathcal{E}_n}\Big(A,B\Big)(D)$.

A straightforward computation also shows that

$$\rho_{U_1,\dots,U_r,D}\Big(\int_{U_1}f,\dots,\int_{U_r}f\Big)=\int_D f.$$

The cochain complexes $U\mapsto RHom_A^{\mathcal{E}_n}\Big(A,B\Big)(U)\cong Hom_{\mathcal{A}(U)}^{\mathcal{E}_n}\Big(\int_U A,\int_U B\Big)$ and the structure maps (29) above assembles into a locally constant factorization algebra over $\mathbb{R}^n$, yielding an E_n-algebra structure to $RHom_A^{\mathcal{E}_n}\big(A,B\big)$. This is the content of the following result:

Theorem 7.7. *Let $f:A\to B$ be a map of E_n-algebras.*

1. *The structure maps (29) $\rho_{U_1,\dots,U_r,V}$ make $U\mapsto RHom_A^{\mathcal{E}_n}\big(A,B\big)\big(U\big)$ a locally constant factorization algebra on $\mathbb{R}^n$ whose global section are naturally equivalent to $RHom_A^{\mathcal{E}_n}\Big(A,B\Big)$.*
2. *In particular $HH_{\mathcal{E}_n}(A,B)\cong RHom_A^{\mathcal{E}_n}\big(A,B\big)$ inherits a natural E_n-algebra structure (with unit given by f).*
3. *Let $g:B\to C$ be another map of E_n-algebras. The (derived) functor of composition of E_n-modules homomorphisms*
$$RHom_A^{\mathcal{E}_n}\big(A,B\big)\otimes RHom_B^{\mathcal{E}_n}\big(B,C\big)\stackrel{\circ}{\longrightarrow}RHom_A^{\mathcal{E}_n}\big(A,C\big)$$
is a homomorphism of E_n-algebras[18].
4. *Let $h:C\to D$ be an E_n-algebra map. The canonical map*
$$RHom_A^{\mathcal{E}_n}\big(A,B\big)\otimes RHom_C^{\mathcal{E}_n}\big(C,D\big)\longrightarrow RHom_{A\otimes C}^{\mathcal{E}_n}\big(A\otimes C,B\otimes D\big)$$
is a homomorphism of E_n-algebras.

[18]the left hand side being endowed with the E_n-algebra structure induced on the tensor products of E_n-algebras and the A-module structure on C being given by the E_n-algebra map $g\circ f:A\to C$

The naturality (in B) of the E_n-algebra structure of $RHom_A^{\mathcal{E}_n}\big(A,B\big)$ means that, given a morphism $\phi : B \to B'$ of E_n-algebras, the induced map

$$\phi_* : RHom_A^{\mathcal{E}_n}\big(A,B\big) \to RHom_A^{\mathcal{E}_n}\big(A,B'\big) \quad \text{(given by } g \mapsto \phi\circ g)$$

is an E_n-algebra morphism. Here the A-module structure of B is of course given by the E_n-algebra morphism $\phi\circ f : A \to B'$. Similarly, the naturality in A means that, given a morphism $\psi : A' \to A$ of E_n-algebras, the induced map

$$\psi_* : RHom_A^{\mathcal{E}_n}\big(A,B\big) \to RHom_{A'}^{\mathcal{E}_n}\big(A',B\big) \quad \text{(given by } g \mapsto g\circ \psi)$$

is an E_n-algebra morphism.

Proof of Theorem 7.7. Since the global section $\mathcal{F}(\mathbb{R}^n)$ of a locally constant factorization algebra $\mathcal{F}$ on $\mathbb{R}^n$ is an E_n-algebra (Proposition 2.15), the second statement is an immediate consequence of the first one. We now prove the first one.

First we prove the naturality of the structure maps (29) with respect to the inclusion of open disks. That is we need to check that for a family of pairwise disjoints disks $U_1,\dots U_r$ inside a disk V and families $W_1^j \dots W_{i_j}^j$ of pairwise disjoints disks inside U_j (for $j = 1\dots r$) we have

$$\rho_{U_1,\dots,U_r,V}\Big(\rho_{W_1^1,\dots,W_{i_1}^1,U_1},\dots,\rho_{W_1^r,\dots,W_{i_r}^r,U_r}\Big) = \rho_{W_1^1,\dots,W_{i_1}^1,\dots,W_1^r,\dots,W_{i_r}^r,V}. \tag{30}$$

For $k\in\{1,\dots,r\}$, let $U_{k,\partial} = U_k \setminus \big(\coprod_{j=1}^{i_k}(W_j^k - col_j^k)\big)$ be the "boundary" of the union of the disks $W_1^k \coprod\cdots\coprod W_{i_k}^k$ in U_k (as constructed in the definition of the map $\rho_{W_1^k,\dots,W_{i_k}^k,U_k}$). By Lemma 7.5, we may assume that the collars $col_k \subset U_k$ does not intersect any W_j^k. By construction, the composition

$$\rho_{U_1,\dots,U_r,V}\Big(\rho_{W_1^1,\dots,W_{i_1}^1,U_1}(g_1^1,\dots,g_{i_1}^1),\dots,\rho_{W_1^r,\dots,W_{i_r}^r,U_r}(g_1^r,\dots,g_{i_r}^r)\Big)$$

is thus given by gluing the maps $g_j^k : \mathcal{A}_{|W_j^k} \to \mathcal{B}_{|W_j^k}$ defined on each small disk W_j^k, the maps $f_{k,\partial} = \int_{U_{k,\partial}} f : \int_{U_{k,\partial}} A \to \int_{U_{k,\partial}} B$ induced by f on each $U_{k,\partial}$, and the map $f_\partial = \int_{U_\partial} f$ where $U_\partial = V \setminus \big(\coprod_{i=1}^r (U_i - col_i)\big)$ (we follow the notation introduced to define the map (29)). Note that the union of U_∂ with the $U_{k,\partial}$ is simply the "boundary" $\tilde{U}_\partial := V \setminus \big(\coprod_{j,k}(W_j^k - col_j^k)\big)$. Further the gluing of the maps $f_{k,\partial}$ and f_∂ is simply the map (induced on the stratified factorization algebra by) $\int_{\tilde{U}_\partial} f : \int_{\tilde{U}_\partial} A \to \int_{\tilde{U}_\partial} B$. The gluing of this latter map with the maps $g_j^k : \mathcal{A}_{|W_j^k} \to \mathcal{B}_{|W_j^k}$ is, by definition, the map

$$\rho_{W_1^1,\dots,W_{i_1}^1,\dots,W_1^r,\dots,W_{i_r}^r,V}(g_1^1,\dots,g_{i_1}^1,\dots,g_1^r,\dots,g_{i_r}^r)$$

which proves the identity (30).

It remains to check that $U \mapsto RHom_A^{\mathcal{E}_n}\Big(A,B\Big)(U)$ is locally constant. Since the factorization algebras $\mathcal{A}$ and $\mathcal{B}$ are locally constant, the natural maps $\int_U A \to \int_V A$ and $\int_U B \to \int_V B$ are equivalences for any embedding $U \hookrightarrow V$ of a disk U inside a bigger disk V. By definition we have

$$\begin{aligned} RHom_A^{\mathcal{E}_n}\Big(A,B\Big)(U) &\cong Hom_{\mathcal{A}(U)}^{\mathcal{E}_n}\Big(\int_U A, \int_U B\Big),\\ RHom_A^{\mathcal{E}_n}\Big(A,B\Big)(V) &\cong Hom_{\mathcal{A}(V)}^{\mathcal{E}_n}\Big(\int_V A, \int_V B\Big). \end{aligned}$$

By definition, for any $g \in RHom_A^{\mathcal{E}_n}\big(A,B\big)\big(U\big)$, the map

$$\rho_{U,V} : Hom^{\mathcal{E}_n}_{\mathcal{A}(U)}\Big(\int_U A, \int_U B\Big) \longrightarrow Hom^{\mathcal{E}_n}_{\mathcal{A}(V)}\Big(\int_V A, \int_V B\Big)$$

applied to g is induced by a map of factorization algebras $\rho_{U,V}(g) : \mathcal{A}_{|V} \to \mathcal{B}_{|V}$ whose restriction to U is just g. It follows that the following diagram is commutative

$$\begin{tikzcd} \int_V A \arrow[r, "\rho_{U,V}(g)"] & \int_V B \\ \int_U A \arrow[u, "\simeq"] \arrow[r, "g"] & \int_U B \arrow[u, "\simeq"'] \end{tikzcd}$$

for all $g \in RHom_A^{\mathcal{E}_n}\big(A,B\big)\big(U\big)$. Since the vertical maps are equivalences and independent of g, it follows that $\rho_{U,V} : Hom^{\mathcal{E}_n}_{\mathcal{A}(U)}\Big(\int_U A, \int_U B\Big) \to Hom^{\mathcal{E}_n}_{\mathcal{A}(V)}\Big(\int_V A, \int_V B\Big)$ is an equivalence. Note in particular that, taking $V = \mathbb{R}^n$, we have canonical equivalences

$$\begin{aligned} RHom_A^{\mathcal{E}_n}\Big(A,B\Big)\big(U\big) &\cong Hom^{\mathcal{E}_n}_{\mathcal{A}(U)}\Big(\int_U A, \int_U B\Big) \\ &\cong Hom^{\mathcal{E}_n}_{\mathcal{A}(\mathbb{R}^n)}\Big(\int_{\mathbb{R}^n} A, \int_{\mathbb{R}^n} B\Big) \cong RHom_A^{\mathcal{E}_n}\Big(A,B\Big) \end{aligned}$$

for any disk U in $\mathbb{R}^n$.

A map of E_n-algebras $g : B \to C$ induces a canonical object in $RHom_B^{\mathcal{E}_n}(B,C)$ given by g itself. Thus the naturality of the E_n-algebra structure (claimed in assertion **(2)**) is in fact a consequence of the assertion **(3)** in the Theorem (that we will prove below). To finish the proof of claims **(1)**, **(2)** in the Theorem we need to see that the canonical element $f \in RHom_A^{\mathcal{E}_n}(A,B)$ is a unit. Indeed, let $U_1, \dots, U_r, V$ be a finite family of pairwise disjoints disks inside a bigger disk D, and $g_i \in RHom_A^{\mathcal{E}_n}(A,B)(U_i)$ $(i = 1\dots r)$. Denote $f \in RHom_A^{\mathcal{E}_n}(A,B)(V)$ the canonical element induced by f.

Let $U_\partial := D \setminus \big(\coprod_{i=1}^r (U_i - col_i)\big)$ be the "boundary" of the opens $\coprod_{i=1}^r U_i$ and $D_\partial := D \setminus \big((V - col_V) \coprod \Big(\coprod_{i=1}^r (U_i - col_i)\big)\Big)$ be the "boundary" of $V \coprod \big(\coprod_{i=1}^r U_i\big)$. Then, by definition, the gluing of the factorization algebra map $f : \mathcal{A}_{|V} \to \mathcal{B}_{|V}$ and $f_\partial : \mathcal{A}_{|D_\partial} \to \mathcal{B}_{|D_\partial}$ is the map $f : \mathcal{A}_{|U_\partial} \to \mathcal{B}_{|U_\partial}$. Thus we get

$$\rho_{U_1,\dots,U_r,V,D}(g_1,\dots,g_r,f) = \rho_{U_1,\dots,U_r,D}(g_1,\dots,g_r)$$

and it follows that f is a unit for the E_n-algebra structure of $RHom_A^{\mathcal{E}_n}(A,B)$.

We now prove statement **(3)**. Since the B-module structure of C is given by the E_n-algebra map $g : B \to C$, the (derived) composition of maps $RHom_A^{\mathcal{E}_n}\big(A,B\big) \otimes RHom_B^{\mathcal{E}_n}\big(B,C\big) \xrightarrow{\circ} Hom_{k\text{-Mod}_\infty}\big(A,C\big)$ naturally lands in $RHom_A^{\mathcal{E}_n}(A,C)$, where C is endowed with the A-module structure induced by the E_n-algebra morphism $g \circ f : A \to C$. Since the tensor product of E_n-algebras is induced by the tensor products of (locally constant) factorization algebras, it remains to prove that, for any family $U_1, \dots, U_r$ of pairwise disjoint open disks included inside a bigger disk

D, the following diagram
(31)

$$\begin{tikzcd}
\bigotimes_{i=1}^{r}\Big(RHom_A^{\mathcal{E}_n}\big(A,B\big)(U_i)\otimes RHom_B^{\mathcal{E}_n}\big(B,C\big)(U_i)\Big) \arrow[r, "\bigotimes_{i=1}^{r}\circ"] \arrow[d, "\rho^{\otimes 2}_{U_1,\dots,U_r,D}"'] & \bigotimes_{i=1}^{r} RHom_A^{\mathcal{E}_n}\big(A,C\big)(U_i) \arrow[d, "\rho_{U_1,\dots,U_r,D}"] \\
RHom_A^{\mathcal{E}_n}\big(A,B\big)(D)\otimes RHom_B^{\mathcal{E}_n}\big(B,C\big)(D) \arrow[r, "\circ"] & RHom_A^{\mathcal{E}_n}\big(A,C\big)(D)
\end{tikzcd}$$

is commutative in k-Mod$_\infty$. Let be given $\phi_i \in RHom_A^{\mathcal{E}_n}\big(A,B\big)(U_i)$ and $\psi_i \in RHom_B^{\mathcal{E}_n}\big(B,C\big)(U_i)$. we keep denoting $\phi_i : \mathcal{A}_{|U_i} \to \mathcal{B}_{|U_i}$ and $\psi_i : \mathcal{B}_{|U_i} \to \mathcal{C}_{|U_i}$ the induced maps of factorization algebras. Let $U_\partial = D \setminus \big(\coprod_{i=1}^r (U_i - col_i)\big)$ denotes the boundary of $\coprod_{i=1}^r U_i$ (as denoted above in the definition of $\rho_{U_1,\dots,U_r,D}$). Then, the composition of the upper map and right vertical map in diagram (31) applied to the tensor product $\bigotimes_{i=1}^r \big(\phi_i \otimes \psi_i\big)$ is the map of factorization algebras obtained by gluing the maps of factorization algebras $\psi_i \circ \phi_i : \mathcal{A}_{|U_i} \to \mathcal{C}_{|U_i}$ (on U_i) and $g \circ f : \mathcal{A}_{|U_\partial} \to \mathcal{C}_{|U_\partial}$ (on $U_{|\partial}$). On the open set U_i, the map

$$\psi_i \circ \phi_i : \mathcal{A}_{|U_i} \longrightarrow \mathcal{C}_{|U_i}$$

is the composition $\mathcal{A}_{|U_i} \xrightarrow{\phi_i} \mathcal{B}_{|U_i} \xrightarrow{\psi_i} \mathcal{C}_{|U_i}$. Then the commutativity of diagram (31) follows since the composition of the left map and lower map in diagram (31) (applied to the tensor product $\bigotimes_{i=1}^r \big(\phi_i \otimes \psi_i\big)$) is the composition of the map given by the gluing of the maps $\psi_i : \mathcal{B}_{|U_i} \to \mathcal{C}_{|U_i}$ $(i = 1 \dots r)$ with $g_\partial : \mathcal{B}_{|U_\partial} \to \mathcal{C}_{|U_\partial}$ on one hand with, on the other hand, the map of factorization algebras obtained by gluing the maps $\phi_i : \mathcal{A}_{|U_i} \to \mathcal{B}_{|U_i}$ $(i = 1 \dots r)$ with $f_\partial : \mathcal{A}_{|U_\partial} \to \mathcal{B}_{|U_\partial}$.

It remains to prove statement (4) in Theorem 7.7 which is almost trivial: let $h : C \to D$ be an E_n-algebra map and denote $\mathcal{C}$, $\mathcal{D}$ the associated factorization algebras on $\mathbb{R}^n$. By [L-HA, Theorem 5.3.3.1],

$$\int_{U\cup V} A \;\cong\; \int_U A \otimes \int_V A$$

for any E_n-algebra A and disjoint open sets U, V. Thus, the factorization algebra associated (in Proposition 6.1) to $A\otimes C$ is given by $U \mapsto \mathcal{A}(U)\otimes\mathcal{C}(U)$. It follows that for any pairwise disjoints open disks $U_1, \dots, U_r$ included in a bigger disk $V \subset D^n$, and maps $g_i \in RHom_A^{\mathcal{E}_n}(A,B)(U_i)$, $g_i' \in RHom_C^{\mathcal{E}_n}(C,D)(U_i)$ $(i = 1 \dots r)$, the map

$$\begin{aligned}\rho_{U_1,\dots,U_r,V}\Big(\big(g_1\otimes\cdots\otimes g_r\big)\otimes\big(g_1'\otimes\cdots\otimes g_r'\big)\Big) &\in \Big(RHom_A^{\mathcal{E}_n}(A,B)\otimes RHom_C^{\mathcal{E}_n}(C,D)\Big)(V)\\ &\cong RHom_A^{\mathcal{E}_n}(A,B)(V)\otimes RHom_C^{\mathcal{E}_n}(C,D)(V)\end{aligned}$$

is the map obtained by gluing the maps $g_i\otimes g_i' : \mathcal{A}_{|U_i}\otimes\mathcal{C}_{|U_i} \to \mathcal{B}_{|U_i}\otimes\mathcal{D}_{|U_i}$ $(i = 1 \dots r)$ and the map $f_\partial \otimes h_\partial : \mathcal{A}_{|U_\partial}\otimes\mathcal{C}_{|U_\partial} \to \mathcal{B}_{|U_\partial}\otimes\mathcal{D}_{|U_\partial}$ (here we use freely the notations introduced to define the structure maps $\rho_{U_1,\dots,U_r,V}$ right before Lemma 7.5). Hence,

this map agrees with the map obtained by evaluating the composition

$$\Big(\bigotimes_{i=!}^{r} RHom_A^{\mathcal{E}_n}(A,B)(U_i)\Big)\otimes\Big(\bigotimes_{i=!}^{r} RHom_C^{\mathcal{E}_n}(C,D)(U_i)\Big)$$
$$\longrightarrow \bigotimes_{i=!}^{r} RHom_{A\otimes C}^{\mathcal{E}_n}(A\otimes C, B\otimes D)(U_i)$$
$$\overset{\rho_{U_1,\dots,U_r,V}}{\longrightarrow}\ RHom_{A\otimes C}^{\mathcal{E}_n}(A\otimes C, B\otimes D)(V)$$

at the tensor product $\big(g_1\otimes\cdots\otimes g_r\big)\otimes\big(g_1'\otimes\cdots\otimes g_r'\big)$. This proves that the canonical maps

$$RHom_A^{\mathcal{E}_n}\big(A,B\big)(V)\otimes RHom_C^{\mathcal{E}_n}\big(C,D\big)(V)\longrightarrow RHom_{A\otimes C}^{\mathcal{E}_n}\big(A\otimes C,B\otimes D\big)(V)$$

assembles into a map of factorization algebras and, consequently,

$$RHom_A^{\mathcal{E}_n}\big(A,B\big)\otimes RHom_C^{\mathcal{E}_n}\big(C,D\big)\longrightarrow RHom_{A\otimes C}^{\mathcal{E}_n}\big(A\otimes C,B\otimes D\big)$$

is a homomorphism of E_n-algebras. □

Remark 7.8. The E_n-algebra structure given by Theorem 7.7 is in fact the solution of a universal property as will be given by Proposition 7.19 below which identifies $HH_{\mathcal{E}_n}(A,B)$ with the centralizer of the map $f: A\to B$.

Example 7.9. Assume $n=1$, then

$$HH_{\mathcal{E}_1}(A,B)\cong RHom_A^{\mathcal{E}_1}\big(A,B\big)\cong RHom_{\int_{S^0} A}\big(A,B\big)\cong RHom_{A\otimes A^{op}}\big(A,B\big)$$

is the standard Hochschild cohomology of the algebra A with value in the algebra B. It is straightforward that the E_1-structure given by Theorem 7.7 is induced on the standard Hochschild complex by the usual cup-product [Ge].

Example 7.10. Assume $A=k$ the ground ring and let $f: k\to B$ be the unit map. We have a canonical equivalence $RHom_k^{\mathcal{E}_n}(k,B)\cong B$ in k-Mod_∞. This equivalence is in fact an *equivalence of E_n-algebras*[19]. Since $f: k\to B$ is the unit map, it is immediate from the definition of the structure maps (29) to check that the locally constant factorization algebra structure of $RHom_k^{\mathcal{E}_n}(k,B)$ is the one of $\mathcal{B}$, the locally constant factorization algebra on $\mathbb{R}^n$ associated to B (in § 2.4).

Remark 7.11. Avoiding choices in Theorem 7.7 : There is a way to choose "collars" in the construction of Theorem 7.7 by using a slight variation of the notion of (locally constant) factorization algebras, *i.e.*, (locally constant) $N(\mathrm{Disk}(M))$-algebras. Following Lurie [L-HA, Remark 5.2.4.8], we let $N(\mathrm{Disk}(M)')$ be the ∞-operad associated to the colored operad $\mathrm{Disk}(M)'$ whose objects are open embeddings $\mathbb{R}^n\hookrightarrow M$ and whose morphisms $\mathrm{Disk}(M)'(\phi,\psi)$ are commutative diagrams

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, "h"] \arrow[dr, "\phi"'] & & \mathbb{R}^n \arrow[dl, "\psi"] \\ & M & \end{tikzcd}$$

where f is an *open embedding*. Note that the obvious functor $\phi\mapsto\phi(\mathbb{R}^n)$ is an equivalence of categories. Hence, we can define a locally constant factorization algebra $\mathcal{F}$ as a rule which associates to each embedding $\phi:\mathbb{R}^n\to M$ a chain

[19] where the left hand side is endowed with the E_n-algebra structure given by Theorem 7.7

complex $\mathcal{F}(\phi)$ with natural maps $\mathcal{F}(\phi_1)\otimes\cdots\otimes\mathcal{F}(\phi_r)\to\mathcal{F}(\psi)$ associated to any open embedding $h:\coprod_{i=1}^r\mathbb{R}^n\to\mathbb{R}^n$ such that $\psi\circ h=\coprod_{i=1}^r\phi_i:\coprod_{i=1}^r\mathbb{R}^n\to M$ (satisfying the obvious associativity and symmetry conditions). Further, for any $h:\phi\mapsto\psi$ (*i.e.* $\psi\circ h=\phi$), the structure map $\mathcal{F}(\phi)\to\mathcal{F}(\psi)$ is required to be a quasi-isomorphism.

Let $f:A\to B$ be an E_n-algebra map. Then we can associate to any open embedding $\phi:\mathbb{R}^n\to\mathbb{R}^n$ a chain complex

$$HH_{\mathcal{E}_n}(A,B)(\phi):=RHom^{\mathcal{E}_n}_{\mathcal{A}(\phi(\mathbb{R}^n))}(A,B)(U)\cong RHom^{left}_{\int_{\partial U}A}\Big(\int_U A,\int_U B\Big) \tag{32}$$

where $\partial U:=\phi\big(\mathbb{R}^n\setminus D(0,1)\big)$ is the image by ϕ of the complement of a closed (bounded) euclidean disk centered at 0. Any different choice of radius yields canonically equivalent chain complexes. Then to any open embedding $h:\coprod_{i=1}^r\mathbb{R}^n\to\mathbb{R}^n$ such that $\psi\circ h=\coprod_{i=1}^r\phi_i:\coprod_{i=1}^r\mathbb{R}^n\to M$, one associates a map

$$HH_{\mathcal{E}_n}(A,B)(\phi_1)\otimes\cdots\otimes HH_{\mathcal{E}_n}(A,B)(\phi_r)\longrightarrow HH_{\mathcal{E}_n}(A,B)(\psi)$$

in the same way as we did in Lemma 7.5, using the canonical collars given by the complement of a disk centered at 0 inside each disk $\mathbb{R}^n$. Then, using this slight alternative definition of factorization algebras, one proves (in a similar way) the obvious analogue of Theorem 7.7. This approach allows to avoid unessential choices in the construction of the structure maps of the factorization algebra on D^n. Further, it can also similarly be applied in the proof of Proposition 7.12.

If $f:A\to B$ is a map of E_∞-algebras, then Theorem 7.7 and Proposition 7.2 give an E_n-algebra structure to $CH^{S^n}(A,B)$. The latter has also an E_n-algebra structure given by Theorem 5.11. The following result shows that these two structures are the same.

Proposition 7.12. *Let $f:A\to B$ be a map of E_∞-algebra and let B be endowed with the induced A-E_∞-module structure. Then the natural equivalence $HH_{\mathcal{E}_n}(A,B)\cong CH^{S^n}(A,B)$ given by Proposition 7.2 is an equivalence of E_n-algebras*[20].

Proof. The proof of Proposition 7.2 shows that we have equivalences

(33)
$$\begin{aligned}HH_{\mathcal{E}_n}(A,B)\cong RHom^{left}_{\int_{S^{n-1}}A}\Big(\int_{D^n}A,B\Big)&\cong RHom^{left}_{CH_{S^{n-1}}(A)}\big(CH_{D^n}(A),B\big)\\&\cong Hom^{\mathcal{E}_n}_{CH_{D^n}(A)}\big(CH_{D^n}(A),B\big).\end{aligned}$$

By Theorem 3.11, we have natural (in spaces and E_∞-algebras) equivalences $\int_U A\cong CH_U(A)$ and, by the value on a point axiom in Definition 3.23, a canonical equivalence $CH_U(B)\cong B$. We can thus define a rule $U\mapsto Hom^{\mathcal{E}_n}_{CH_U(A)}\big(CH_U(A),B\big)$ and structure maps (for $U_1,\dots,U_r$ pairwise disjoints disk included in a disk D)

$$\begin{aligned}\rho_{U_1,\dots,U_r,D}:Hom^{\mathcal{E}_n}_{CH_{U_1}(A)}\big(CH_{U_1}(A),B\big)\otimes\cdots\otimes Hom^{\mathcal{E}_n}_{CH_{U_r}(A)}\big(CH_{U_r}(A),B\big)&\\ \longrightarrow Hom^{\mathcal{E}_n}_{CH_D(A)}\big(CH_D(A),B\big)&\end{aligned} \tag{34}$$

[20]where the left hand-side is the E_n-algebra given by Theorem 7.7 and the right hand side is the E_n-algebra given by Theorem 5.11

defined exactly as the structure maps (29) for $RHom_A^{\mathcal{E}_n}(A,B)(U)$. Then the proof of Theorem 7.7 applies *mutatis mutandis* to prove that $U \mapsto Hom^{\mathcal{E}_n}_{CH_U(A)}\big(CH_U(A),B\big)$ is a *locally constant factorization algebra on* D^n and further, that the equivalences

$$Hom^{\mathcal{E}_n}_{\mathcal{A}(U)}\Big(\int_U A, \int_U B\Big) \;\cong\; Hom^{\mathcal{E}_n}_{CH_U(A)}\big(CH_U(A),B\big)$$

(induced by Theorem 3.11) are an equivalence of factorization algebras.

Now, for any collar $\partial U \cong S^{n-1}\times(0,\epsilon)$ inside a disk U, we have natural equivalences

$$\begin{aligned}
Hom^{\mathcal{E}_n}_{CH_U(A)}\big(CH_U(A),B\big) &\cong RHom^{left}_{CH_{\partial U}(A)}\big(CH_U(A),B\big)\\
&\cong RHom_A\Big(A, RHom^{left}_{CH_{\partial U}(A)}\big(CH_U(A),B\big)\Big)\\
&\cong RHom_A\Big(CH_U(A) \overset{\mathbb{L}}{\underset{CH_{\partial U}(A)}{\otimes}} A,\, B\Big)\\
&\cong RHom_A\Big(CH_{U/\partial U}(A),B\big) \cong CH^{U/\partial U}(A,B)
\end{aligned}$$

where the last equivalences are by the excision axiom (Definition 3.23) and definition of Hochschild cochains. Recall from the proof of Proposition 7.2 that, for $U = D^n$, the above equivalences and the equivalences (33) are precisely the natural equivalence $HH_{\mathcal{E}_n}(A,B)\cong CH^{S^n}(A,B)$ of Proposition 7.2. Hence, we are left to prove Proposition 7.12 replacing $HH_{\mathcal{E}_n}(A,B)$ with $CH^{S^n}(A,B)$ endowed with the E_n-algebra structure given by the locally constant factorization algebra structure.

By functoriality of Hochschild chains, we have natural maps of E_∞-algebras

$$\bigotimes_{i=1}^r B \cong \bigotimes_{i=1}^r CH_{U_i}(B) \cong CH_{\bigcup_{i=1}^r U_i}(B) \overset{(\bigcup_{i=1}^r U_i\hookrightarrow V)_*}{\longrightarrow} CH_V(B)\cong B \tag{35}$$

Further, we have pinching maps $D \overset{p_{U_1,\dots,U_r,D}}{\longrightarrow} \bigvee_{i=1}^r \big(U_i/\partial U_i\big)$ obtained by collapsing $D\setminus\big(\bigcup_{i=1}^r U_i\setminus\partial U_i\big)$ to a point. By functoriality of Hochschild cochains, the pinching maps yield a map

$$\bigotimes_{i=1}^r CH_{U_i}(A) \cong CH_{\bigcup_{i=1}^r U_i}(A) \overset{(p_{U_1,\dots,U_r,D})_*}{\longrightarrow} CH_{\bigvee_{i=1}^r \big(U_i/\partial U_i\big)}(A). \tag{36}$$

Using the last two maps, we get the composition

$$\begin{aligned}
\bigotimes_{i=1}^r RHom^{left}_{CH_{\partial U_i}(A)}\big(CH_{U_i}(A),B\big) &\longrightarrow RHom_{\bigotimes_{i=1}^r A}\big(\bigotimes_{i=1}^r CH_{U_i}(A), \bigotimes_{i=1}^r B\big) \\
\longrightarrow RHom_{\bigotimes_{i=1}^r A}\big(\bigotimes_{i=1}^r CH_{U_i}(A),\, B\big) &\cong RHom_A\big(CH_{\bigvee_{i=1}^r \big(U_i/\partial U_i\big)}(A),\, B\big)\\
&\longrightarrow RHom_A\big(CH_V(A),\, B\big)
\end{aligned} \tag{37}$$

where the last maps are respectively induced by the map (35), Lemma 5.2 and the map (36). Using the homotopy invariance of Hochschild cochains and unfolding the definition of $\rho_{U_1,\dots,U_r,D}$, we find that the structure map (34) is transfered to the above map (37) under the natural equivalences $RHom^{left}_{CH_{\partial U}(A)}\big(CH_U(A),B\big)\cong$

$CH^{U/\partial U}(A,B)$ (where U is any disk in D^n). Note that when the U_i are cubes in $\mathcal{C}_n(r)$ and D is D^n, the composition of the map (37) with the equivalence

$$\Big(\bigotimes_{i=1}^{r} CH^{S^n}(A,B)\Big) \cong \bigotimes_{i=1}^{r} RHom^{left}_{CH_{S^{n-1}}(A,B)}\big(CH_{D^n}(A),B\big)$$
$$\cong \bigotimes_{i=1}^{r} RHom^{left}_{CH_{\partial U_i}(A)}\big(CH_{U_i}(A),B\big)$$

is the pinching map (20) $pinch^*_{S^d,r}(c) : \Big(CH^{S^d}(A,B)\Big)^{\otimes r} \longrightarrow CH^{S^d}(A,B)$ (where c is the cube associated to the U_i's). Now, thanks to Theorem 5.11 and the definition of the factorization algebra structure on $U \mapsto Hom^{\mathcal{E}_n}_{CH_U(A)}\big(CH_U(A),B\big)$, we can apply Lemma 7.13 below which implies that the two E_n-algebra structure on $CH^{S^n}(A,B)$ (given by Theorem 5.11 and the one introduced in this proof by the structures maps (37)) are equivalent. $\square$

Lemma 7.13. *Let $A \in k\text{-}Mod_\infty$ and assume that*

(1) *A has an $\mathcal{C}_n$-algebra structure,* i.e.*, an E_n-algebra structure given by an action of the chains little n-dimensional cube operad.*
(2) *There is a locally constant factorization algebra $\mathcal{A}$ on $\mathbb{R}^n$ (identified with the open unit cube) together with an equivalence $\varphi : \mathcal{A}(\mathbb{R}^n) \xrightarrow{\simeq} A$ (in $k\text{-}Mod_\infty$); which thus induces another E_n-algebra structure on A.*

Assume further that the two structures given by (1) *and* (2) *are* compatible *in the following sense: for any configuration of cubes $c \in \mathcal{C}_n(r)$, the following diagram*

(38)
$$\begin{tikzcd}
A^{\otimes r} \arrow[rr, "\mu_c"] & & A \\
\mathcal{A}(c_1)\otimes\cdots\otimes\mathcal{A}(c_r) \arrow[u, "\bigotimes_{i=1}^r \varphi\circ\rho_{c_i,\mathbb{R}^n}"] \arrow[rr, "\rho_{c_1,\dots,c_r,\mathbb{R}^n}"] & & \mathcal{A}(\mathbb{R}^n)\,, \arrow[u, "\varphi"']
\end{tikzcd}$$

(where we denote $c_1,\dots,c_r$ the configuration of cubes defined by $c = (c_1,\dots,c_r)$, μ_c is the structure map given by the operadic structure and $\rho_{U_1,\dots,U_r,V}$ the structure maps of the factorization algebra structure), is commutative in $k\text{-}Mod_\infty$.

Then the two E_n-algebras structures on A defined by (1) *and* (2) *are equivalent (in E_n-Alg).*

Proof. The E_n-algebra structure defined by (1), that is by the action of the little n-dimensional cube operad on A yields a locally factorization algebra $\mathcal{A}'$ on $\mathbb{R}^n$ which is equivalent to the E_n-structure given by (1) and satisfies $\mathcal{A}'(U) \cong \int_U A$ (see Proposition 2.15 and Theorem 2.14). Thus we only have to prove that $\mathcal{A}'$ is equivalent to $\mathcal{A}$ as a factorization algebra and thus to analyze further the construction of $\mathcal{A}'$.

The $\mathcal{C}_n$-action on A gives a structure of $\mathbb{E}^{\otimes}_{\mathbb{R}^n}$-algebra to A, where $\mathbb{E}^{\otimes}_{\mathbb{R}^n}$ is the ∞-operad introduced by Lurie in [L-HA, Section 5.2], that is, the operad whose algebras are precisely those given by Definition 2.3. The canonical map of operad $\mathcal{C}_n \to \mathbb{E}^{\otimes}_{\mathbb{R}^n}$ is an equivalence by the results of Lurie [L-HA, Example 5.2.4.3]; further we can replace $\mathcal{C}_n$ by the operad of little rectangles. Hence we get an $\mathbb{E}^{\otimes}_{\mathbb{R}^n}$-algebra structure on A.

By [L-HA, Theorem 5.2.4.9] (also see § 2.4), the ∞-operad map $N(Disk(\mathbb{R}^n))^{\otimes} \to \mathbb{E}^{\otimes}_{\mathbb{R}^n}$ now yields the locally constant factorization algebra structure on $\mathbb{R}^n$ (denoted $\mathcal{A}'$ above). By construction, we have natural (with respect to the factorization algebra structure) equivalences $\mathcal{A}'(c_i) \stackrel{\simeq}{\to} A$ for any open cube $c \subset \mathbb{R}^n$.

Since the family of open cubes inside $\mathbb{R}^n$ forms a factorization basis of $\mathbb{R}^n$, it is enough to check that the factorization algebra structures on $\mathcal{A}'$ and $\mathcal{A}$ are equivalent on these basis of open cubes, which form a factorizing cover (see [CG] for the definition of factorization algebra on a basis of open subsets and the fact that a factorization algebra is determined by its value on a basis). This is precisely the compatibility condition (38) of the Lemma and the result follows. ☐

Remark 7.14. It is possible, though more technically involved, to use directly, in the spirit of Section 5.2, the little cube operad $\mathcal{C}_n$ to make $RHom^{\mathcal{E}_n}\Big(A,B\Big)$ an E_n-algebra. We now sketch how to do this, leaving to the interested reader the task to fill in the many details.

We let again $\mathcal{A}$, $\mathcal{B}$ be the factorization algebras corresponding to A, B. Recall that we have factorizations algebras $\mathcal{A}^{\otimes k}$, $\mathcal{B}^{\otimes k}$ on $\coprod_{i=1}^{k} D^n$ and similarly for $\mathcal{B}$. Let $c \in \mathcal{C}_n(r)$ be a framed embedding $\coprod_{i=1}^{r} D^n \hookrightarrow D^n$. Then the little cube c induces a natural (in $\mathcal{A}$ and c) equivalence $\mathcal{A}_{|c^{-1}(D^n)} \cong \mathcal{A}^{\otimes r}$. We can define a map

$$comp_r(f,c) : RHom^{\mathcal{E}_n}\Big(A,B\Big)^{\otimes r} \longrightarrow RHom^{\mathcal{E}_n}\Big(A,B\Big)$$

similarly to the definition of the structure maps (29). Indeed, we first use c to see the factorization algebras $\mathcal{A}$ and $\mathcal{B}$ on D^n as obtained by gluing together $r+1$-factorization algebras on D^n. The image $c(\coprod_{i=1}^{r} D^n)$ has r-open connected components, denoted $D_1,\dots,D_r$. Choosing small collars $col_1,\dots col_r$ in the neighborhood of each $D^n \subset \coprod_{i=1}^{r} D^n$ yield a connected open set $U_\partial := D^n \setminus c\big(\coprod_{i=1}^{r}(D^n - col_i)\big)$. Since c is an embedding, it induces an identification $\mathcal{A} \cong \mathcal{A}_{|D_i}$ for each $i = 1\dots r$. Thus from any family of maps $g_1,\dots,g_r \in RHom^{\mathcal{E}_n}\big(\mathcal{A},\mathcal{B}\big)$, we get induced maps of factorization algebras $g_i : \mathcal{A}_{|D_i} \to \mathcal{B}_{|D_i}$. Further, we also have, by restriction of f to the open set U_c, an induced map $f_c : \mathcal{A}_{|U_c} \to \mathcal{B}_{|U_c}$.

The argument of the proof of Lemma 7.5 apply to show

Lemma 7.15. *The family $(g_1,\dots,g_r,f_c)$ of maps of factorization algebras glues together to define a map of factorization algebras*

$$comp_r(f,c)\,(g_1,\dots,g_r) \in Hom(\mathcal{A},\mathcal{B})$$

Further, on global sections, the induced map $comp_r(f,c)\,(g_1,\dots,g_r)\big(D^n\big) : A \to B$ is a map of A-E_n-module.

It follows (from the above Lemma 7.5) that we have a well-defined map $(g_1,\dots,g_r) \mapsto comp_r(f,c)(g_1,\dots,g_r)(D^n)$, simply denoted by

$$comp_r(f,c) : RHom^{\mathcal{E}_n}\Big(A,B\Big)^{\otimes r} \longrightarrow RHom^{\mathcal{E}_n}\Big(A,B\Big). \tag{39}$$

Recall that the set of A-E_n-modules homomorphisms is simplicially enriched. Similarly, there are simplicial sets of maps of factorization algebras, see [CG]. Equivalently, we have topological spaces of such maps. Using the fact that the factorization algebras $\mathcal{A}$ and $\mathcal{B}$ are locally constant, one can prove the following

Lemma 7.16. *The map* $comp_r(f,c) : RHom^{\mathcal{E}_n}\Big(A,B\Big)^{\otimes r} \longrightarrow RHom^{\mathcal{E}_n}\Big(A,B\Big)$ *depends continuously on* c.

The above Lemma 7.16 allows to consider the maps $comp_r(f,c)$ in families over the operad of little cubes and thus one can let c runs through the operad $\mathcal{C}_n(r)$ so that we get the first part of the following result.

Proposition 7.17. *Let* $f : A \to B$ *be a map of* E_n*-algebras.*

(1) *The maps* $comp_r(f,c)$ *assembles to give a map*

$$comp_r(f) : C_*\big(\mathcal{C}_n(r)\big) \otimes RHom^{\mathcal{E}_n}\Big(A,B\Big)^{\otimes r} \longrightarrow RHom^{\mathcal{E}_n}\Big(A,B\Big)$$

in k-Mod_∞.

(2) *The maps* $comp_r(f)$ *gives to* $RHom^{\mathcal{E}_n}\Big(A,B\Big)$ *a natural* E_n*-algebra structure.*

The proof of the second assertion of this Proposition is essentially the same as the ones of Theorem 7.7 and Theorem 5.11.

7.3. **E_n-Hochschild cohomology as centralizers.** We will now relate the natural E_n-algebra structure of $RHom_A^{\mathcal{E}_n}(A,B)$ (for an E_n-algebra map $f : A \to B$) given in Section 7.2 with the centralizer $\mathfrak{z}(f)$. The following definition is due to Lurie [L-HA, Lu3] (and generalize the notion of center of a category due to Drinfeld).

Definition 7.18. The (derived) centralizer of an E_n-algebra map $f : A \to B$ is the *universal* E_n-algebra $\mathfrak{z}(f)$ endowed with a homomorphism of E_n-algebras $e_{\mathfrak{z}(f)} : A \otimes \mathfrak{z}(f) \to B$ making the following diagram

$$\begin{tikzcd} & A \otimes \mathfrak{z}(f) \arrow[dr, "e_{\mathfrak{z}(f)}"] & \\ A \arrow[ur, "id\otimes 1_{\mathfrak{z}(f)}"] \arrow[rr, "f"] & & B \end{tikzcd} \tag{40}$$

commutative in E_n-Alg.

The existence of the derived centralizer $\mathfrak{z}(f)$ of an E_n-algebra map $f : A \to B$ is a *non-trivial* Theorem of Lurie [L-HA, Lu3]. The universal property of the centralizer implies that there are natural maps of E_n-algebras

$$\mathfrak{z}(\circ) : \mathfrak{z}(f) \otimes \mathfrak{z}(g) \longrightarrow \mathfrak{z}(g \circ f) \tag{41}$$

see [L-HA, Lu3].

The E_n-algebra structure on the $\mathcal{E}_n$-Hochschild cohomology given by Theorem 7.7 gives an explicit description of the centralizer $\mathfrak{z}(f)$ (as an E_n-algebra):

Proposition 7.19. *Let* $f : A \to B$ *be an* E_n*-algebra map and endow* $HH_{\mathcal{E}_n}(A,B)$ *with the* E_n*-algebra structure given by Theorem 7.7.*

Then the $\mathcal{E}_n$*-Hochschild cohomology* $HH_{\mathcal{E}_n}(A,B) \cong RHom_A^{\mathcal{E}_n}\big(A,B\big)$ *is the centralizer* $\mathfrak{z}(f)$, i.e., *there is a natural equivalence of* E_n*-algebras* $HH_{\mathcal{E}_n}(A,B) \cong \mathfrak{z}(f)$

such that, for any E_n-algebra map $g : B \to C$, the following diagram

$$\begin{tikzcd} RHom_A^{\mathcal{E}_n}\big(A,B\big) \otimes RHom_B^{\mathcal{E}_n}\big(B,C\big) \arrow[r, "\cong\otimes\cong"] \arrow[d, "\circ"'] & \mathfrak{z}(f)\otimes\mathfrak{z}(g) \arrow[d, "\mathfrak{z}(\circ)"] \\ RHom_A^{\mathcal{E}_n}\big(A,C\big) \arrow[r, "\cong"] & \mathfrak{z}(g\circ f) \end{tikzcd}$$

commutes in E_n-Alg.

Remark 7.20. Note that in the proof of Proposition 7.19, we *do not* assume the existence of centralizers, but actually prove that $HH_{\mathcal{E}_n}(A,B)$ satisfies the universal property of centralizers. In particular the proof of Proposition 7.19 implies the existence of centralizers of any map $f : A \to B$ of E_n-algebras.

We first prove a lemma. Denote $ev : A \otimes RHom_A^{\mathcal{E}_n}\big(A,B\big) \to B$ the (derived) evaluation map $(a,f) \mapsto f(a)$.

Lemma 7.21. *The evaluation map $ev : A \otimes RHom_A^{\mathcal{E}_n}\big(A,B\big) \to B$ is an E_n-algebra morphism. Further, the following diagram*

$$\begin{tikzcd} & A\otimes RHom_A^{\mathcal{E}_n}\big(A,B\big) \arrow[dr, "ev"] & \\ A \arrow[ur, "id\otimes 1"] \arrow[rr, "f"] & & B \end{tikzcd}$$

is commutative in E_n-Alg.

Proof. There are canonical equivalences of E_n-algebras $RHom_k^{\mathcal{E}_n}(k,A) \cong A$ and $RHom_k^{\mathcal{E}_n}(k,B) \cong B$ (see Example 7.10). Thus, the fact that ev is a map of E_n-algebras follows from statement (3) in Theorem 7.7. Further, the same Theorem implies that the unit of $RHom_A^{\mathcal{E}_n}\big(A,B\big)$ is $f : A \to B$. It follows that $ev\circ(id\otimes 1) = f$ which proves the Lemma. ☐

Proof of Proposition 7.19. By Lemma 7.21, we already know that $HH^{\bullet}_{\mathcal{E}_n}(A,B) \cong RHom_A^{\mathcal{E}_n}(A,B)$ fits into a commutative diagram similar to diagram (42) below. We have to prove that for any E_n-algebra $\mathfrak{z}$, endowed with a E_n-algebra map $\phi : A \otimes \mathfrak{z} \to B$ fitting in a commutative diagram

$$\begin{tikzcd} & A\otimes\mathfrak{z} \arrow[dr, "\phi"] & \\ A \arrow[ur, "id\otimes 1_{\mathfrak{z}}"] \arrow[rr, "f"] & & B, \end{tikzcd} \tag{42}$$

there exists an E_n-algebra map $\mathfrak{z} \to RHom_A^{\mathcal{E}_n}$ which makes $A \otimes \mathfrak{z} \xrightarrow{\phi} B$ factor through $A \otimes RHom_A^{\mathcal{E}_n}\big(A,B\big) \xrightarrow{ev} B$.

Let $\theta_\phi : \mathfrak{z} \to RHom(A,B)$ be the map associated to $\phi : A \otimes \mathfrak{z} \to B$ under the (derived) adjunction $RHom(A\otimes\mathfrak{z},B) \cong RHom\big(\mathfrak{z}, RHom(A,B)\big)$ (in k-Mod$_\infty$).

We now prove that θ_ϕ takes values in $RHom_A^{\mathcal{E}_n}(A,B)$. We use again the factorization algebra characterization of E_n-algebras. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{Z}$ be the locally constant factorization algebras associated to A, B and $\mathfrak{z}$. For any open sub-disk

$D \hookrightarrow D^n$, we get the induced map[21]

$$\phi : \big(\mathcal{A}\otimes\mathcal{Z}\big)(D) \cong \int_D A \otimes \int_D \mathfrak{z} \xrightarrow{\int_D \phi} \int_D B \cong \mathcal{B}(D)$$

and its (derived) adjoint $\theta_\phi : \mathcal{Z}(D) \longrightarrow RHom(\mathcal{A}(D), \mathcal{B}(D))$. We are left to check that this last map is compatible with the factorization algebra structures (describing the A-E_n-module structure of A and B). Let $U_0, U_1, \dots, U_r$ be pairwise disjoints open disks included in a bigger disk V, where we assume that U_0 contains the base point of D^n. Also we use the same notation

$$\rho_{U_0,\dots,U_r,V} : \mathcal{F}(U_0)\otimes\cdots\otimes\mathcal{F}(U_r) \longrightarrow \mathcal{F}(V)$$

for the associated structure maps of any one of the factorization algebras $\mathcal{F} = \mathcal{A}$, $\mathcal{B}$ or $\mathcal{Z}$ on D^n. Since $\phi : A\otimes\mathfrak{z} \to B$ is a map of E_n-algebras, for any $a_i \in \mathcal{A}(U_i)$ $(i = 1\dots r)$, $x \in \mathcal{A}(U_0)$ and $z \in \mathfrak{z}(U_0)$, we have

$$\begin{aligned}
\phi\Big(\rho_{U_0,\dots,U_r,V}\big(x, a_1,\dots,a_r\big)\otimes\rho_{U_0,V}\big(z\big)\Big) &= \phi\Big(\rho_{U_0,\dots,U_r,V}\big(x\otimes z, a_1\otimes 1_{\mathfrak{z}},\dots,a_r\otimes 1_{\mathfrak{z}}\big)\Big)\\
&= \rho_{U_0,\dots,U_r,V}\Big(\phi(x\otimes z), \phi(a_1\otimes 1_{\mathfrak{z}}),\dots\\
&\qquad\qquad \dots,\phi(a_r\otimes 1_{\mathfrak{z}})\Big)\\
&= \rho_{U_0,\dots,U_r,V}\big(\phi(x\otimes z), f(a_1),\dots,f(a_r)\big)
\end{aligned}$$

where the last identity follows from the commutativity of diagram (42). Note that the map $z \mapsto \rho_{U_0,V}\big(z\big)$ is an equivalence (since $\mathcal{Z}$ is locally constant). Since the A-E_n-module structure on B is given by f, the above string of equalities ensures that θ_ϕ is a map from $\mathfrak{z}$ to $RHom_A^{\mathcal{E}_n}(A,B)$. In particular, the map $\theta_\phi : \mathfrak{z} \to RHom(A,B)$ factors as

$$\mathfrak{z} \xrightarrow{\tilde{\theta_\phi}} RHom_A^{\mathcal{E}_n}(A,B) \cong RHom^{left}_{\int_{S^{n-1}} A}\Big(\int_{D^n} A, \int_{D^n} B\Big) \hookrightarrow RHom(A,B).$$

To finish the proof of Proposition 7.19, we need to check that $\tilde{\theta_\phi} : \mathfrak{z} \to RHom_A^{\mathcal{E}_n}(A,B)$ is a map of E_n-algebras. Recall that there are equivalences $\mathfrak{z} \cong RHom_k^{\mathcal{E}_n}(k,\mathfrak{z})$, $\mathfrak{z} \cong k \cong \mathfrak{z}$ of E_n-algebras (see Example 7.10). By definition of the (derived) adjunction, $\tilde{\theta_\phi} : \mathfrak{z} \to RHom_A^{\mathcal{E}_n}(A,B)$ factors as the composition

(43)
$$\begin{aligned}
\mathfrak{z} \cong k\otimes\mathfrak{z} \xrightarrow{1_{RHom_A^{\mathcal{E}_n}(A,A)}\otimes id} RHom_A^{\mathcal{E}_n}(A,A)\otimes\mathfrak{z} &\cong RHom_A^{\mathcal{E}_n}(A,A)\otimes RHom_k^{\mathcal{E}_n}(k,\mathfrak{z})\\
&\longrightarrow RHom_A^{\mathcal{E}_n}(A, A\otimes\mathfrak{z}) \xrightarrow{\phi_*} RHom_A^{\mathcal{E}_n}(A,B).
\end{aligned}$$

By Theorem 7.7.(2) and (4), the last two maps are an E_n-algebra Homomorphisms. Thus the composition (43) is a composition of E_n-algebras maps hence $\tilde{\theta_\phi} : \mathfrak{z} \to RHom_A^{\mathcal{E}_n}(A,B)$ itself is a map of E_n-algebras.

Further, by definition of θ_ϕ, the identity

$$ev\circ\big(id_A\otimes\theta_\phi\big) \;=\; \phi$$

[21]we make, for simplicity, an abuse of notation by still denoting ϕ the induced map and similarly with θ_ϕ below

holds. Hence we eventually get a commutative diagram

$$\begin{tikzcd}
A \arrow[dr, "id\otimes 1_{\mathfrak{z}}"] \arrow[drrr, "id\otimes 1_{RHom_A^{\mathcal{E}_n}(A,B)}"] \arrow[ddrr, bend right, "f"'] & & & \\
& A\otimes \mathfrak{z} \arrow[rr, "id\otimes\tilde{\theta_\phi}"] \arrow[dr, "\phi"] & & A\otimes RHom_A^{\mathcal{E}_n}(A,B) \arrow[dl, "ev"] \\
& & B &
\end{tikzcd}$$

in E_n-Alg.

It remains to prove the uniqueness of the map $\mathfrak{z}\to RHom_A^{\mathcal{E}_n}(A,B)$ inducing such a commutative diagram. Thus assume that $\alpha:\mathfrak{z}\to RHom_A^{\mathcal{E}_n}(A,B)$ is a map of E_n-algebras such that the following diagram

(44)
$$\begin{tikzcd}
A \arrow[dr, "id\otimes 1_{\mathfrak{z}}"] \arrow[drrr, "id\otimes 1_{RHom_A^{\mathcal{E}_n}(A,B)}"] \arrow[ddrr, bend right, "f"'] & & & \\
& A\otimes \mathfrak{z} \arrow[rr, "id\otimes\alpha"] \arrow[dr, "\phi"] & & A\otimes RHom_A^{\mathcal{E}_n}(A,B) \arrow[dl, "ev"] \\
& & B &
\end{tikzcd}$$

is commutative in E_n-Alg. Note that the composition

$$\begin{aligned}
(45)\quad RHom_A^{\mathcal{E}_n}(A,B) &\cong RHom_k^{\mathcal{E}_n}\Big(k, RHom_A^{\mathcal{E}_n}(A,B)\Big) \\
&\xrightarrow{1_{RHom_A^{\mathcal{E}_n}(A,A)}\otimes id} RHom_A^{\mathcal{E}_n}(A,A)\otimes\Big(k, RHom_A^{\mathcal{E}_n}(A,B)\Big) \\
&\longrightarrow RHom_A^{\mathcal{E}_n}\Big(A, A\otimes RHom_A^{\mathcal{E}_n}(A,B)\Big) \\
&\xrightarrow{ev_*} RHom_A^{\mathcal{E}_n}(A,B)
\end{aligned}$$

is the identy map. From the commutativity of Diagram (44), we get the following commutative diagram

(46)
$$\begin{tikzcd}
\mathfrak{z}\cong RHom_k^{\mathcal{E}_n}(k,\mathfrak{z}) \arrow[r, "\alpha_*"] \arrow[d] & RHom_A^{\mathcal{E}_n}(A,B)\cong RHom_k^{\mathcal{E}_n}\big(k, RHom_A^{\mathcal{E}_n}(A,B)\big) \arrow[d] \\
RHom_A^{\mathcal{E}_n}(A, A\otimes\mathfrak{z}) \arrow[dr, "\phi_*"] & RHom_A^{\mathcal{E}_n}\big(A, A\otimes RHom_A^{\mathcal{E}_n}(A,B)\big) \arrow[d, "ev_*"] \\
& RHom_A^{\mathcal{E}_n}(A,B)
\end{tikzcd}$$

in E_n-Alg. The composition of the right vertical maps in Diagram (46) is the composition (45), hence is the identity, and the upper map is $\alpha:\mathfrak{z}\to RHom_A^{\mathcal{E}_n}(A,B)$. It follows that the map α is equivalent to the map (43) hence to $\tilde{\theta_\phi}$. This gives the uniqueness statement and the Proposition will follow once we proved the diagram

depicted in Proposition 7.19 is commutative. The latter is an immediate consequence of the universal property of the centralizers (and thus of $HH_A^{\mathcal{E}_n}(A,B)$) and of Theorem 7.7.(3). $\square$

Example 7.22. Assume $B = k$ so that $f : A \to k$ is an augmentation. Then by Proposition 7.19 and [L-HA, Example 6.1.4.14] and [Lu-MP, Remark 7.13] there is an equivalence of E_n-algebras

$$HH_{\mathcal{E}_n}(A,k) \cong RHom(Bar^{(n)}(A),k)$$

where $Bar^{(n)}(A)$ is the E_n-coalgebra given by the iterated Bar construction on A, that is, the (derived) *Koszul dual* of A. Thus Theorem 7.7 gives an explicit description of the E_n-algebra structure on the dual of $Bar^{(n)}(A)$. See Section 9 for a more detailed description.

Combining Proposition 7.19 and Proposition 7.12, we get

Corollary 7.23. *let $f : A \to B$ be a map of E_∞-algebras. Then the Hochschild cochains $CH^{S^n}(A,B)$ over the n-sphere (endowed with its E_n-algebra structure given by Theorem 5.11) is the centralizer $\mathfrak{z}(f)$ of f viewed as a map of E_n-algebras (by restriction).*

Remark 7.24. Assume $f : A \to B$ and $g : B \to C$ are maps of CDGA's. Then by the above Corollary 7.23 or Proposition 7.12, there is a composition

$$CH^{S^n}(A,B) \otimes CH^{S^n}(B,C) \xrightarrow{\circ} CH^{S^n}(A,C) \tag{47}$$

(which is a map of E_n-algebras) induced by the natural equivalence $CH^{S^n}(A,B) \cong RHom^{left}_{CH_{S^{n-1}}(A)}(CH_{D^n}(A),B)$ and (derived) compositions of homomorphisms. In the setting of CDGA's, this composition can be represented in an easy way as follows. Let $I_\bullet$ be the standard simplicial model of the interval ([G, GTZ]); its boundary $\partial I^n_\bullet$ is a simplicial model for S^{n-1}. Then the map (47) is represented by the usual composition (of left dg-modules)

$$\begin{aligned} Hom^{left}_{CH_{\partial I^n_\bullet}(A)}\big(CH_{I^n_\bullet}(A), CH_{I^n_\bullet}(B)\big) \otimes Hom^{left}_{CH_{\partial I^n_\bullet}(B)}\big(CH_{I^n_\bullet}(B), CH_{I^n_\bullet}(C)\big) \\ \xrightarrow{\circ} Hom^{left}_{CH_{\partial I^n_\bullet}(A)}\big(CH_{I^n_\bullet}(A), CH_{I^n_\bullet}(C)\big) \end{aligned}$$

since $CH_{I^n_\bullet}(A)$ is a (semi-)free $CH_{\partial I^n_\bullet}(A)$-algebra.

7.4. **Higher Deligne conjecture.** In this section we deal with (some of) the solutions of higher Deligne conjecture. That is we specialized the results of the previous sections 7 and 5.2 to the case $A = B$ and $f = id$.

By Theorem 7.7 above, the composition of morphisms of A-E_n-modules

$$RHom_A^{\mathcal{E}_n}(A,A) \otimes RHom_A^{\mathcal{E}_n}(A,A) \xrightarrow{\circ} RHom_A^{\mathcal{E}_n}(A,A) \tag{48}$$

is a homomorphism of E_n-algebras (with unit given by the identity map $id : A \to A$). The composition of morphisms is further (homotopy) associative and unital (with unit id); thus $RHom_A^{\mathcal{E}_n}(A,A)$ is actually an E_1-algebra in the ∞-category E_n-Alg.

By the ∞-category version of Dunn Theorem [Du, L-HA, Lu3] or see Theorem 2.16, there is an equivalence of $(\infty,1)$-categories $E_1 - Alg\big(E_n - Alg\big) \cong E_{n+1} -$

Alg. Thus the multiplication (48) lift the E_n-algebra structure of $HH^{\bullet}_{\mathcal{E}_n}(A, A) \cong RHom^{\mathcal{E}_n}_A(A, A)$ to an E_{n+1}-algebra structure.

In particular we just proved the first part of the following result, which has already been given by Francis [F1] (and Lurie [L-HA, Lu3]).

Theorem 7.25. (Higher Deligne Conjecture)

(1) *Let A be an E_n-algebra. There is a natural E_{n+1}-algebra structure on $HH^{\bullet}_{\mathcal{E}_n}(A, A)$ with underlying E_n-algebra structure given by Theorem 7.7*[22].
(2) *Let now A be an E_∞-algebra. Then there is a natural E_{n+1}-algebra structure on $CH^{S^n}(A, A)$ whose underlying E_n-algebra structure is the one given by Theorem 5.11. In particular, the underlying E_1-algebra structure is given by the standard cup-product (see Corollary 5.6 and Example 5.7).*
(3) *For A an E_∞-algebra, the two E_{n+1}-structures given by statements (1) and (2) are equivalent.*

Proof. We have already proved the first claim. Note that the underlying E_n-algebra structure of an E_{n+1}-algebra is induced by the pushforward along the canonical projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, see Theorem 2.16. By Proposition 7.2, $CH^{S^n}(A, A)$ also inherits a structure of E_{n+1}-algebra whose underlying E_n-algebra is the same as the one given by Theorem 5.11 thanks to Proposition 7.12. This proves both claims (2) and (3). □

Example 7.26. In the case $n = 1$, Theorem 7.25 recovers the original Deligne conjecture asserting the existence of a natural E_2-algebra structure on the Hochschild cochains lifting the associative algebra structure induced by the cup-product. It can be proved that this E_2-algebra structure induces the usual Gerstenhaber algebra structure (from [Ge]) on the Hochschild cohomology groups.

Remark 7.27. Francis [F1] has given a different solution to the higher Deligne conjecture. His solution is directly and explicitly related to the degree n Lie algebra structure on $HH_{\mathcal{E}_n}(A, A)$. However, the underlying cup-product (*i.e.* E_1-algebra structure) is more mysterious. This is the contrary with the solution given by Theorem 7.25. This latter solution is, by definition, the same as the one of Lurie [L-HA]. It would therefore be very interesting and useful to relate Francis construction with ours. Note that the explicit knowledge of the cup-product is useful to us to relate this construction to the higher string topology operations, see § 8.

8. Integral chain models for higher string topology operations

We will use the E_∞-Poincaré duality and Hochschild chains to give an algebraic model for Brane Topology at the *chain* level, over an arbitrary coefficient ring.

8.1. **Brane operations for n-connected Poincaré duality space.** Recall that the n-dimensional free sphere space is denoted $X^{S^n} = Map(S^n, X)$. It is the space of continuous map from S^n to X endowed with the compact-open topology. Sullivan and Voronov [CV, Section 5] have shown that there is a natural graded commutative algebra structure, called the *sphere product*, on the shifted homology $H_{\bullet+\dim(M)}(M^{S^n})$ of an oriented closed manifold. For $n = 1$, this structure agrees with Chas-Sullivan loop product [CS]. This product was extended to all oriented stacks in [BGNX]. For $n = 2$, the sphere product is a special case of the surface

[22] where we take $B = A$ and $f = id$

product studied in [GTZ]. Further, it is claimed that $H_{\bullet+\dim(M)}(M^{S^n})$ is an algebra over the homology $H_*(\mathcal{E}^{fr}_{n+1})$ of the framed little disk operad $\mathcal{E}^{fr}_{n+1}$. Below we will forget about the $SO(n+1)$-action and deal with action of the $\mathcal{E}_{n+1}$-operad at the *chain* (and not homology) level and without specific assumptions on the characteristic of the ground ring k.

We start by stating one of our main result:

Theorem 8.1. *Let X be a n-connected Poincaré duality space whose homology groups are projective k-modules. Then the shifted chain complex $C_{\bullet+\dim(X)}(X^{S^n})$ has a natural*[23] *E_{n+1}-algebra structure which induces the sphere product* [CV, Section 5]

$$H_p\big(X^{S^n}\big)\otimes H_q\big(X^{S^n}\big)\to H_{p+q-\dim(X)}\big(X^{S^n}\big)$$

in homology when X is an oriented closed manifold.

Proof. Remark 6.18 and the assumptions on the homology groups of X implies that the homology groups of X are projective finitely generated so that the biduality homomorphism $C_*(X)\to(C^*(X))^\vee$ is a quasi-isomorphism. Since X is a Poincaré duality space, it then follows from Corollary 6.19 that the Poincaré duality map (25)

$$\chi_X : C^*(X)\to C_*(X)[\dim(X)] \;\cong\; \big(C^*(X)\big)^\vee[\dim(X)]$$

is an equivalence of $C^*(X)$-E_∞-Modules. Thus it yields an equivalence

$$\begin{aligned}(49)\quad CH^{S^n}\left(C^*(X),C^*(X)\right) &\cong Hom_{C^*(X)}\left(CH_{S^n}(C^*(X)),C^*(X)\right)\\ &\xrightarrow{(\chi_X)\circ -} Hom_{C^*(X)}\Big(CH_{S^n}(C^*(X)),\big(C^*(X)\big)^\vee\Big)[\dim(X)]\\ &\cong CH^{S^n}\Big(C^*(X)),\big(C^*(X)\big)^\vee\Big)[\dim(X)].\end{aligned}$$

Since X is n-connected with projective finitely generated homology groups, by Corollary 4.7, there is an equivalence

$$CH^{S^n}\Big(C^*(X)),\big(C^*(X)\big)^\vee\Big) \;\cong\; C_*\big(X^{S^n}\big). \qquad (50)$$

Combining the equivalences (49) and (50), we get a natural equivalence

$$CH^{S^n}\left(C^*(X),C^*(X)\right)\cong C_*\big(X^{S^n}\big)[\dim(X)]. \qquad (51)$$

By Theorem 7.25, $CH^{S^n}\left(C^*(X),C^*(X)\right)$ has a natural E_{n+1}-algebra structure, whose underlying E_1-algebra structure is given by the cup-product. Hence the equivalence (51) yields a natural E_{n+1}-structure on $C_*\big(X^{S^n}\big)[\dim(X)]$. Note that the naturality with respect to maps $f:X\to Y$ of Poincaré duality spaces follows from Theorem 8.8 below since a Poincaré duality space yields an object of $\mathcal{AM}$ and a map of Poincaré duality space is a map in $\mathcal{AM}$, see Example 8.6.(2) below. From this observation follows the commutativity of the following diagram (in which

[23]with respect to maps of Poincaré duality spaces in the sense of Definition 6.20

$d = \dim(X) = \dim(Y)$)

$$\begin{tikzcd}
\Big(CH^{S^n}\left(C^*(X), C^*(X)\right)\Big)^{\otimes 2} \arrow[r, "\circ"] \arrow[d, "(\chi_X)\circ -^{\otimes 2}"', "\cong"] & CH^{S^n}\left(C^*(X), C^*(X)\right) \arrow[d, "\cong"', "(\chi_X)\circ -"] \\
\Big(CH^{S^n}\left(C^*(X), C_*(X)\right)[d]\Big)^{\otimes 2} \arrow[d, "(f_*)^{\otimes 2}"'] & CH^{S^n}\left(C^*(X), C_*(X)\right)[d] \arrow[d, "f_*"] \\
\Big(CH^{S^n}\left(C^*(Y), C_*(Y)\right)[d]\Big)^{\otimes 2} & CH^{S^n}\left(C^*(Y), C_*(Y)\right)[d] \\
\Big(CH^{S^n}\left(C^*(Y), C^*(Y)\right)\Big)^{\otimes 2} \arrow[r, "\circ"] \arrow[u, "(\chi_Y)\circ -^{\otimes 2}", "\cong"'] & CH^{S^n}\left(C^*(Y), C^*(Y)\right) \arrow[u, "\cong", "(\chi_Y)\circ -"']
\end{tikzcd}$$

where the horizontal arrows are given by the composition (48) of (derived) homomorphisms (and Proposition 7.2). By Theorem 8.8, the vertical maps are maps of E_n-algebras. Thus the above diagram shows that a map of Poincaré duality space induces a map of E_1-algebras (with respect to the composition (48)) in the (symmetric monoidal) category of E_n-algebras and thus induces a map of E_{n+1}-algebras by Dunn Theorem (see [Du, L-HA] or Theorem 2.16): $E_1 - Alg\big(E_n - Alg\big) \cong E_{n+1} - Alg$.

It remains to identify the underlying multiplication in homology with its purely topological counterpart. This is done in Section 8.2, see Proposition 8.15. □

Example 8.2. The assumptions on the projectivity of the homology groups are automatic when k is a field. In particular theorem 8.1 applies to $C_{\bullet+\dim(M)}(M^{S^n}, k)$ for any closed oriented manifold M and $k = \mathbb{Q}$ or $k = \mathbb{F}_p$ a finite field.

Example 8.3. Assume M is a simply connected closed manifold (with projective homology groups). Then Theorem 8.1 yields an E_2-structure on the chains $C_*(LM)[\dim(M)]$ of the free loop space LM, thus string topology operations at the chain level. According to Example 7.26 and Proposition 8.15 below, the underlying Gerstenhaber structure is the classical Chas-Sullivan one [CS].

Corollary 8.4. *Let X, Y be n-connected ($n \geq 1$) closed manifolds of same dimension and assume $f : M \to N$ induces an isomorphism in homology such that $f_*([X]) = [Y] \in H_*(Y, k)$. Then the induced bijection $H_*(X^{S^n}) \cong H_*(Y^{S^n})$ is an algebra isomorphism (with respect to the sphere product).*

In particular, the sphere product is an homotopy invariant of n-connected manifolds (with respect to orientation preserving maps).

Proof. By assumption, the induced map $\cap f_*([X]) : C^*(Y) \to C_*(Y)[\dim(Y)]$ and $\cap[Y]C^*(Y) \to C_*(Y)[\dim(Y)]$ are homotopic. Thus f induces a map of Poincaré duality spaces $(X, [X]) \to (Y, [Y])$ which is a quasi-isomorphism. Then, by Theorem 8.1, $f_* : C_{*+\dim(X)}(X^{S^n}) \to C_{*+\dim(Y)}(Y^{S^n})$ is an equivalence of E_n-algebras. In particular, it is an algebra isomorphism in homology so that the result follows from the identification of the sphere product as asserted in Theorem 8.1 (see Proposition 8.15). □

The above brane product fits into a larger setting of setups[24] to define $\mathcal{E}_n$-actions on $CH^{S^n}(A,M)$. In fact, we start with the following general setup.

Definition 8.5. We define $\mathcal{AM}$ as the following category. The objects of $\mathcal{AM}$ are triples (A,M,μ), where A is an E_∞-algebra, M is an E_∞-A-module, and, considering the E_∞-algebra $A\otimes A$ with canonical E_∞-$(A\otimes A)$-modules M and $M\otimes M$ (induced via the E_∞ structure map $A\otimes A\to A$), we assume that $\mu : M\otimes M\to M$ is an E_∞-$(A\otimes A)$-module map[25]. The morphisms of $\mathcal{AM}$ consist of tuples $(f,g):(A,M,\mu)\to(A',M',\mu')$, where $f:A\to A'$ is an E_∞-morphism, thus inducing an E_∞-A-module structure on M', and $g:M'\to M$ is an E_∞-A-module map, satisfying the compatibility relation,

$$\begin{tikzcd} M'\otimes M' \arrow[r, "\mu'"] \arrow[d, "g\otimes g"'] & M' \arrow[d, "g"] \\ M\otimes M \arrow[r, "\mu"] & M \end{tikzcd} \tag{52}$$

in k-Mod_∞.

There are two main examples we have in mind for the above definition.

Example 8.6.
(1) The first example relates to the sphere product as considered in Section 5.2 and also in [G]. Let A and B be two E_∞-algebras, and let $h:A\to B$ be a morphism of E_∞-algebras. Then, h makes $M:=B$ into an E_∞-A-module, and the E_∞ structure of B gives a map $B\otimes B\to B$ which is also an E_∞-$(A\otimes A)$-module map. Furthermore, if h factors through an E_∞-algebra B' as a composition of E_∞-algebras maps $h:A\stackrel{h'}{\to}B'\stackrel{g}{\to}B$, then this induces a morphism between the spaces $(id_A,g):(A,B,\mu)\to(A,B',\mu')$.
(2) The second example relates to generalizations of sphere topology products as described in Theorem 8.1 above. Let A be an E_∞-algebra and M be an E_∞-A-module and given an E_∞-module map $\rho:M\to A$. We define the induced E_∞-$(A\otimes A)$-module map $\mu:M\otimes M\to M$ as the composition of ρ and the E_∞-A-module structure of M,
$$\mu: M\otimes M\stackrel{\rho\otimes id}{\longrightarrow}A\otimes M\longrightarrow M.$$
Furthermore, any map of two given E_∞-A-modules $g:M'\to M$ which commutes with E_∞-A-module maps ρ and ρ',
$$\begin{tikzcd} M' \arrow[dr, "\rho'"] \arrow[dd, "g"'] & \\ & A \\ M \arrow[ur, "\rho"'] & \end{tikzcd}$$

[24]which is useful to study functoriality of brane operations

[25]said otherwise, the objects of $\mathcal{AM}$ are the objects N of the monoidal ∞-category Mod^{E_n} endowed with a structure map $\mu_N : N\otimes N\to N$; the morphisms are however different

also respects the induced relation (52), since $g\circ\mu'(m_1', m_2') = g(\rho'(m_1').m_2') = \rho'(m_1').g(m_2') = \rho(g((m_1')).g(m_2') = \mu\circ(g\otimes g)(m_1', m_2')$.

For example, consider the setup from Section 6.3: $C_*(X)$ is an E_∞-coalgebra, $C^*(X) = Hom_k(C_*(X), k)$ is its linear dual endowed with its canonical E_∞-algebra structure, and caping with the fundamental cycle $\cap[X] : C^*(X) \to C_*(X)[\dim(X)]$ induces an E_∞-quasi-isomorphism of E_∞-A-modules. The quasi-inverse of this map is an E_∞-A-module map $\rho : M := C_*(X)[\dim(X)] \to A := C^*(X)$. Moreover, if $f : (X, [X]) \to (Y, [Y])$ is a map of Poincaré duality space (Definition 6.20), then the tuple $(f^* : C^*(Y) \to C^*(X), f_* : C_*(X) \to C_*(Y))$ is a map in the category $\mathcal{AM}$.

For any triple (A, M, μ) which is an object of $\mathcal{AM}$ described in Definition 8.5, we can consider the Hochschild cochains $CH^{S^d}(A, M)$. We claim that there is an E_d-algebra structure on $CH^{S^d}(A, M)$, generalizing the E_d-algebra structure from Theorem 5.11.

Definition 8.7. Using the notation from Section 5.2, we define the E_d-algebra structure on $CH^{S^d}(A, M)$ by,

$$\begin{aligned}
C_*\big(\mathcal{C}_d(r)\big)\otimes\Big(CH^{S^d}(A,M)\Big)^{\otimes r} &\longrightarrow C_*\big(\mathcal{C}_d(r)\big)\otimes\Big(Hom_A(A^{\otimes S^d}, M)\Big)^{\otimes r}\\
&\longrightarrow C_*\big(\mathcal{C}_d(r)\big)\otimes Hom_{A^{\otimes r}}((A^{\otimes S^d})^{\otimes r}, M^{\otimes r})\\
&\stackrel{id\otimes(\mu^{\circ(r-1)})_*}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big)\otimes Hom_{A^{\otimes r}}((A^{\otimes S^d})^{\otimes r}, M)\\
&\stackrel{\cong}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big)\otimes Hom_{A^{\otimes r}}((A^{\otimes S^d})^{\otimes r}, Hom_A(A,M))\\
&\stackrel{\cong}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big)\otimes Hom_A(A\otimes^{\mathbb{L}}_{A^{\otimes r}}(A^{\otimes S^d})^{\otimes r}, M)\\
&\stackrel{\cong}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big)\otimes Hom_A(A^{\otimes(\overbrace{S^d\vee\cdots\vee S^d}^{r\text{ times}})}, M)\\
&\longrightarrow C_*\big(\mathcal{C}_d(r)\big)\otimes CH^{S^d\vee\cdots\vee S^d}(A,M)\stackrel{pinch^*}{\longrightarrow} CH^{S^d}(A,M).
\end{aligned}$$

We need to show compatibility of the involved operad action. This is similar to the proof in section 5.2.

In fact, more is true:

Theorem 8.8. *The identification given in the previous Definition 8.7 defines a (contravariant) functor $CH^{S^d} : \mathcal{AM} \to E_d - Alg$.*

Proof. It only remains to show that morphisms $(f, g) : (A, M, \mu) \to (A', M', \mu')$ in $\mathcal{AM}$ induce maps of E_d-algebras. Since $f : A \to A'$ makes M' into an E_∞-A-algebra, and with this $\mu' : M'\otimes M' \to M'$ into a map of E_∞-$(A\otimes A)$-modules, this

follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\left(CH^{S^d}(A',M')\right)^{\otimes r} \arrow[r, "(f^*)^{\otimes r}"] \arrow[d] & \left(CH^{S^d}(A,M')\right)^{\otimes r} \arrow[r, "(g_*)^{\otimes r}"] \arrow[d] & \left(CH^{S^d}(A,M)\right)^{\otimes r} \arrow[d] \\
Hom_{A'^{\otimes r}}((A'^{\otimes S^d})^{\otimes r}, M'^{\otimes r}) \arrow[r, "(f^{\otimes r})^*"] \arrow[d, "(\mu'^{\circ(r-1)})_*"'] & Hom_{A^{\otimes r}}((A^{\otimes S^d})^{\otimes r}, M'^{\otimes r}) \arrow[r, "(g^{\otimes r})_*"] \arrow[d, "(\mu'^{\circ(r-1)})_*"] & Hom_{A^{\otimes r}}((A^{\otimes S^d})^{\otimes r}, M^{\otimes r}) \arrow[d, "(\mu^{\circ(r-1)})_*"] \\
Hom_{A'^{\otimes r}}\left((A'^{\otimes S^d})^{\otimes r}, M'\right) \arrow[r, "(f^{\otimes r})^*"] \arrow[d] & Hom_{A^{\otimes r}}\left((A^{\otimes S^d})^{\otimes r}, M'\right) \arrow[r, "g_*"] \arrow[d] & Hom_{A^{\otimes r}}\left((A^{\otimes S^d})^{\otimes r}, M\right) \arrow[d] \\
Hom_{A'}\left(A'^{\otimes(S^d\vee\cdots\vee S^d)}, M'\right) \arrow[r, "f^*"] \arrow[d, "pinch^*"'] & Hom_{A}\left(A^{\otimes(S^d\vee\cdots\vee S^d)}, M'\right) \arrow[r, "g_*"] \arrow[d, "pinch^*"'] & Hom_{A}\left(A^{\otimes(S^d\vee\cdots\vee S^d)}, M\right) \arrow[d, "pinch^*"'] \\
Hom_{A'}(A'^{\otimes S^d}, M') \arrow[r, "f^*"] & Hom_A(A^{\otimes S^d}, M') \arrow[r, "g_*"] & Hom_A(A^{\otimes S^d}, M)
\end{tikzcd}$$

□

By the virtue of the previous theorem and Example 8.6(2), we can thus define a family of sphere topology operations, one for each E_∞-module map $C_*(X)[\dim(X)] \to C^*(X)$, which are related by morphisms of E_d-algebras.

In particular, for $d = 1$, we can obtain (chain level, characteristic free) *string topology* operations associated to any E_∞-module map $C_*(M)[\dim(M)] \to C^*(M)$.

8.2. **Topological identification of the Brane product.** In this section, we prove that the cup product of Hochschild cochains over spheres identifies with the usual "brane product" in the homology of a free sphere space. The idea of the proof follows the surface product kind of proof from [GTZ, Theorem 3.4.2].

We start by recalling the construction of the sphere product of Sullivan-Voronov [CV]. Let M be a manifold equipped with a Riemannian metric and let the sphere spaces $Map(S^n, M)$ be equipped with Fréchet manifold structures. We further assume that M is closed, oriented. We have a cartesian square of fibrations

$$\begin{tikzcd}
Map(S^n \vee S^n, M) \arrow[r, "\rho_{in}"] \arrow[d] & Map(S^n, M) \times Map(S^n, M) \arrow[d, "ev\times ev"] \\
M \arrow[r, "\text{diagonal}"] & M \times M
\end{tikzcd} \tag{53}$$

where the evaluation maps on the right are furthermore submersions. We denote $Tub(M) \subset M \times M$ a tubular neighborhood of the diagonal of M, which can be identified to the normal bundle of the diagonal. The pullback $(ev\times ev)^{-1}(Tub(M))$ by the submersion $ev \times ev : Map(S^n, M) \times Map(S^n, M) \to M \times M$ can be identified with a tubular neighborhood $Tub(Map(S^n \vee S^n, M))$ of ρ_{in} and thus with a normal bundle of ρ_{in}. One forms the corresponding Thom spaces M^{-TM} and $Map(S^n \vee S^n, M)^{-TM}$ by collapsing all the complements of the tubular neighborhood to a point. These Thom spaces are spheres (of dimension $\dim(M)$) bundles

over, respectively M, and $Map(S^n \vee S^n, M)$. Hence, we have a diagram of pullback squares

$$\begin{tikzcd} Map(S^n \coprod S^n, M) \arrow[r, "\text{collapse}"] \arrow[d, "ev\times ev"'] & Map(S^n \vee S^n, M)^{-TM} \arrow[r, "\pi"] \arrow[d, "ev"] & Map(S^n \vee S^n, M) \arrow[d, "ev"] \\ M \times M \arrow[r, "\text{collapse}"] & M^{-TM} \arrow[r, "\pi"] & M \end{tikzcd}$$

where the vertical arrows are fibrations. In particular, the Thom class of ρ_{in} is the pullback $(ev^*)(th(M)) \in H^{\dim(M)}(Map(S^n \vee S^n, M)^{-TM})$ of the Thom class $th(M) \in H^{\dim(M)}(M^{-TM})$ of $M \to M \times M$.

The above setup allows to define a Gysin map

$$(\rho_{in})_! : H_*\Big(M^{S^n \amalg S^n}\Big) \longrightarrow H_{*-\dim(M)}\Big(M^{S^n \vee S^n}\Big)$$

as the composition

$$(\rho_{in})_! = \pi_* \circ (- \cap ev^*(th(M))) \circ (\text{collapse})_*. \tag{54}$$

Definition 8.9 (Sullivan-Voronov [CV])**.** The sphere product is the composition

$$\star_{S^n} : H_{*+\dim(M)}\Big(M^{S^n}\Big)^{\otimes 2} \to H_{*+2\dim(M)}\Big(M^{S^n \amalg S^n}\Big) \stackrel{(\rho_{in})_!}{\longrightarrow} H_{*+\dim(M)}\Big(M^{S^n \vee S^n}\Big) \stackrel{(\delta^*_{S^n})_*}{\longrightarrow} H_{*+\dim(M)}\Big(M^{S^n}\Big)$$

where $\delta_{S^n} : S^n \to S^n \vee S^n$ is the pinching map.

Note that the Thom class $th(M)$ can be represented by any cocycle $t(M)$ which is Poincaré dual to the pushforward of the fundamental cycle $[M]$ of M, *i.e.*, $\chi_{M\times M}(\text{collapse}_*(t(M)) = \big(\text{diagonal}_*([M])\big)$ or, equivalently,

$$\chi_{M^{-TM}}(t(M)) = \big(\text{collapse} \circ \text{diagonal}_*([M])\big).$$

By Corollary 6.15, we get maps of E_∞-modules

$$\rho_{th(M)} : C_*(M^{-TM}) \longrightarrow C_{*-\dim(M)}(M^{-TM}),$$

$$\rho_{ev^*(th(M)} : C_*\big(Map(S^n \vee S^n, M)^{-TM}\big) \longrightarrow C_{*-\dim(M)}\big(Map(S^n \vee S^n, M)^{-TM}\big)$$

lifting the cap-products $-\cap t(M)$ and $-\cap ev^*(t(M))$. Thus we obtain the following chain level interpretation of the sphere product.

Lemma 8.10. *The sphere product (Definition 8.9) is induced by passing to the homology groups in the following composition*

$$\begin{aligned} \star_{S^n} : \big(C_*\big(M^{S^n}\big)[\dim(M)]\big)^{\otimes 2} &\to C_*\big(M^{S^n \amalg S^n}\big)[2\dim(M)] \\ \stackrel{collapse_*}{\longrightarrow} C_*\big(\big(M^{S^n\vee S^n}\big)^{-TM}\big)[2\dim(M)] &\stackrel{\rho_{ev^*(th(M)}}{\longrightarrow} C_*\big(\big(M^{S^n\vee S^n}\big)^{-TM}\big)[\dim(M)] \\ &\stackrel{\pi_*}{\longrightarrow} C_*\big(\big(M^{S^n\vee S^n}\big)[\dim(M)] \stackrel{(\delta^*_{S^n})_*}{\longrightarrow} C_*\big(\big(M^{S^n}\big)[\dim(M)]. \end{aligned} \tag{55}$$

Remark 8.11. In this section we only identify the sphere product which is the degree 0-component of a higher framed E_{n+1}-structure claimed in [CV, Section 5]. The reason is that we do not know higher degree representative of this operations (in a way similar to the map (55)) since such higher operations would involve a careful analysis of Gysin maps associated to higher cacti *in families*. However, it is

possible that the new operads introduced by Bargheer in [Ba] could lead in a near future to explicit representatives of the degree n Lie Bracket in homology.

We now further assume X is a general Poincaré duality space (see Definition 6.16).

Recall that by Corollary 4.7 and Corollary 6.15, we have the equivalence (51):

$$CH^{S^n}(C^*(X), C^*(X)) \;\cong\; C_*\big(X^{S^n}\big)[\dim(X)].$$

The cup-product can be thus transfered (through the above equivalence) to give a multiplication $\Big(C_*\big(X^{S^n}\big)[\dim(X)]\Big)^{\otimes 2} \to C_*\big(X^{S^n}\big)[\dim(X)]$. We first wish to give another chain level representative for this multiplication, which is essentially the content of Lemma 8.13 below. We will then compare it with the sphere product $\star_{S^n}$ given by the composition (55).

The E_∞-algebra map $C^*(X\times X) \stackrel{diag^*}{\longrightarrow} C^*(X)$ induced by the diagonal $X \to X\times X$ makes $C_*(X)$ an E_∞-$C^*(X\times X)$-module. By functoriality of the cap-product, the diagonal $C_*(X) \to C_*(X\times X)$ is a map of left $(C^*(X\times X), \cup)$-module.

By Theorem 6.6, we thus get a unique lift $C_*(X) \stackrel{diag_*}{\longrightarrow} C_*(X\times X)$ of the diagonal map in $C_*(X\times X)$-Mod^{E_∞}. By Lemma 4.1, there is an equivalence of E_∞-algebras $C^*(X\times X) \cong C^*(X)\otimes C^*(X)$. Further, Poincaré duality (Corollary 6.19) gives equivalences of E_∞-$C^*(Y)$-modules $\chi_X : C^*(Y) \stackrel{\simeq}{\to} C_*(Y)[\dim(Y)]$ for any Poincaré duality space Y.

Putting together the last three statements we obtain the first assertion in

Lemma 8.12. *Let X be a Poincaré duality space. There is a map in $C^*(X)\otimes C^*(X) - Mod^{E_\infty}$ given by the following composition:*

$$\begin{aligned}\nabla_X : C^*(X) \stackrel{\simeq}{\to} C_*(X)[\dim(X)] &\stackrel{diag_*}{\longrightarrow} C_*(X\times X)[\dim(X)]\\ &\stackrel{\simeq}{\to} C_*(X\times X)[\dim(X)] \stackrel{\simeq}{\leftarrow} C^*(X\times X)[-\dim(X)]\\ &\qquad\qquad\cong C^*(X)\otimes C^*(X)[-\dim(X)].\end{aligned}$$

Further, for any closed oriented manifold M, the following diagram is commutative

$$\begin{tikzcd} C^*(M) \arrow[r, "\pi^*"] \arrow[drrrr, "\nabla_M"'] & C^*(M^{-TM}) \arrow[r, "\rho^\vee_{th(M)}"] & C^*(M^{-TM})[-\dim(M)] \arrow[rr, "collapse^*"] & & C^*(M\times M)[-\dim(M)] \arrow[d, "\cong"] \\ & & & & C^*(M)\otimes C^*(M)[-\dim(M)] \end{tikzcd}$$

in $C^*(X)\otimes C^*(X) - Mod^{E_\infty}$.

Proof. The second assertion follows from the identity

$$\begin{aligned}\mathrm{collapse}^*\big(\pi^*(x)\big)\cap\big(\mathrm{collapse}^*\big(t(M)\big)\cap[M\times M]\big) &= \mathrm{collapse}^*\big(\pi^*(x)\big)\cap \mathrm{diagonal}_*([M])\\ &= \mathrm{diagonal}_*(x\cap[M])\end{aligned}$$

which follows from $\pi\circ\mathrm{collapse}\circ\mathrm{diagonal} = id$ and the definition of the Thom class. $\square$

It follows that the map ∇_X yields a map of $C^*(X)$-E_∞-modules

$$\begin{aligned}(56)\quad \nabla_{X*}: CH_{S^n\vee S^n}(C^*(X)) &\cong CH_{S^n\coprod S^n}(C^*(X)) \overset{\mathbb{L}}{\underset{C^*(X)^{\otimes 2}}{\otimes}} C^*(X)\\ &\xrightarrow{1\otimes\nabla_X} CH_{S^n\coprod S^n}(C^*(X)) \overset{\mathbb{L}}{\underset{C^*(X)^{\otimes 2}}{\otimes}} C^*(X)^{\otimes 2}[-\dim(X)]\\ &\cong CH_{S^n\coprod S^n}(C^*(X))[-\dim(X)].\end{aligned}$$

Thus, dualizing, we get a map

(57)

$$\begin{aligned}\nabla^! : CH^{S^n\coprod S^n}\Big(C^*(X),(C^*(X))^\vee\Big) &\cong Hom_{C^*(X)}\Big(CH_{S^n\coprod S^n}(C^*(X)),(C^*(X))^\vee\Big)\\ &\xrightarrow{\nabla_{X*}^*} Hom_{C^*(X)}\Big(CH_{S^n\vee S^n}(C^*(X)),(C^*(X))^\vee\Big)[-\dim(X)]\\ &\cong CH^{S^n\vee S^n}\Big(C^*(X),(C^*(X))^\vee\Big)[-\dim(X)].\end{aligned}$$

Recall that we have a pinching map $\delta_{S^n}: S^n\to S^n\vee S^n$ induced by collapsing the equator of S^n to a point. This gives us a multiplication

(58)

$$\begin{aligned}\mu_{S^n}: CH^{S^n}\Big(C^*(X),(C^*(X))^\vee\Big)^{\otimes 2} &\cong Hom_{C^*(X)}\Big(CH_{S^n}(C^*(X)),(C^*(X))^\vee\Big)^{\otimes 2}\\ &\longrightarrow Hom_{C^*(X)^{\otimes 2}}\Big(CH_{S^n\coprod S^n}(C^*(X)),(C^*(X))^\vee\otimes(C^*(X))^\vee\Big)\\ &\cong Hom_{C^*(X)}\Big(CH_{S^n\coprod S^n}(C^*(X)),(C^*(X))^\vee\Big)\\ &\xrightarrow{\nabla^!} CH^{S^n\vee S^n}\Big(C^*(X),(C^*(X))^\vee\Big)[-\dim(X)]\\ &\xrightarrow{\delta_{S^n}^*} CH^{S^n}\Big(C^*(X),(C^*(X))^\vee\Big)[-\dim(X)].\end{aligned}$$

Lemma 8.13. *Let X be a Poincaré duality space. There is a commutative (in k-Mod_∞) diagram*

$$\begin{tikzcd} CH^{S^n}\left(C^*(X),C^*(X)\right)^{\otimes 2} \arrow[r, "\cup_{S^n}"] \arrow[d, "\cong"'] & CH^{S^n}\left(C^*(X),C^*(X)\right) \arrow[d, "\cong"] \\ \left(CH^{S^n}\left(C^*(X),(C^*(X))^\vee\right)[\dim(X)]\right)^{\otimes 2} \arrow[r, "\mu_{S^n}"] & CH^{S^n}\left(C^*(X),(C^*(X))^\vee\right)[\dim(X)] \end{tikzcd}$$

where the top arrow is the sphere cup-product of Corollary 5.6 and the vertical arrows are induced by the Poincaré duality map $\chi_X: C^*(X)\to C_*(X)[\dim(X)]\to \big(C^*(X)\big)^\vee[\dim(X)]$.

Proof. By Lemma 4.1, the E_∞-algebra map $m_X: C^*(X)\otimes C^*(X)\to C^*(X)$ is the composition

$$m_X: C^*(X)\otimes C^*(X)\cong C^*(X\times X)\xrightarrow{diag^*} C^*(X).$$

It follows that the map ∇_X defined in Lemma 8.12 sits inside a commutative diagram

$$\begin{tikzcd} C^*(X)\otimes C^*(X) \arrow[r, "m_X"] \arrow[d, "\chi_X\otimes\chi_X"'] & C^*(X) \arrow[d, "\chi_X"] \\ \big(C^*(X)\big)^\vee \otimes \big(C^*(X)\big)^\vee[2\dim(X)] \arrow[r, "(\nabla_X)^\vee"] & \big(C^*(X)\big)^\vee[\dim(X)] \end{tikzcd}$$

in $C^*(X)\otimes C^*(X)-Mod^{E_\infty}$. It follows that we get a commutative diagram

(59)
$$\begin{tikzcd} Hom_{C^*(X)^{\otimes 2}}\Big(CH_{S^n\coprod S^n}(C^*(X)), C^*(X)^{\otimes 2}\Big) \arrow[r, "(m_X)_*"] \arrow[d, "(\chi_X)_*^{\otimes 2}"'] & Hom_{C^*(X)^{\otimes 2}}\Big(CH_{S^n\coprod S^n}(C^*(X)), C^*(X)\Big) \arrow[d, "(\chi_X)_*"] \\ Hom_{C^*(X)^{\otimes 2}}\Big(CH_{S^n\coprod S^n}(C^*(X)), (C^*(X))^\vee)^{\otimes 2}\Big) \arrow[r, "(\nabla_X)_*^\vee"] \arrow[d, "\cong"'] & Hom_{C^*(X)^{\otimes 2}}\Big(CH_{S^n\coprod S^n}(C^*(X)), (C^*(X))^\vee\Big) \arrow[d, "\cong"] \\ Hom_{C^*(X)}\Big(CH_{S^n\coprod S^n}(C^*(X)), (C^*(X))^\vee\Big) \arrow[r, "\nabla^!"] & Hom_{C^*(X)}\Big(CH_{S^n\vee S^n}(C^*(X)), (C^*(X))^\vee\Big) \end{tikzcd}$$

in k-Mod$_\infty$ (note that we have suppress the degree shifting in the diagram for simplicity). By functoriality, we have a commutative diagram

$$\begin{tikzcd} Hom_{C^*(X)}\Big(CH_{S^n\vee S^n}(C^*(X)), C^*(X)\Big) \arrow[r, "\delta_{S^n}^*"] \arrow[d, "(\chi_X)_*"'] & Hom_{C^*(X)}\Big(CH_{S^n}(C^*(X)), C^*(X)\Big) \arrow[d, "(\chi_X)_*"] \\ Hom_{C^*(X)}\Big(CH_{S^n\vee S^n}(C^*(X)), (C^*(X))^\vee\Big) \arrow[r, "\delta_{S^n}^*"] & Hom_{C^*(X)}\Big(CH_{S^n}(C^*(X)), (C^*(X))^\vee\Big) \end{tikzcd}$$

which, together with the previous diagram (59) and the definition of the map $\cup_{S^n}$ (see Corollary 5.6) and the map (58), implies the Lemma. $\square$

The cartesian square of fibrations (53) shows that, when M is n-connected (and thus M^{S^n} is path connected), there is a quasi-isomorphism

$$C^*\big(M^{S^n\vee S^n}\big) \cong C^*\big(M^{S^n\coprod S^n}\big) \overset{\mathbb{L}}{\underset{C^*\left(M\times M\right)}{\otimes}} C^*(M) \tag{60}$$

so that the map ∇_M of Lemma 8.12 yields a map

$$id\otimes\nabla_M : C^*\big(M^{S^n\vee S^n}\big) \cong C^*\big(M\times M\big) \overset{\mathbb{L}}{\underset{C^*\left(M\times M\right)}{\otimes}} C^*(M)$$
$$\xrightarrow{id\otimes_{C^*\left(M\times M\right)}\nabla_M} C^*\big(M\times M\big)[-\dim(M)] \cong C^*\big(M^{S^n}\big)^{\otimes 2}[-\dim(M)].$$

Lemma 8.14. *Let X be a n-connected Poincaré duality space. The following diagram*

$$\begin{tikzcd}
CH_{S^n}\big(C^*(X)\big) \arrow[r, "(\delta_{S^n})_*"] \arrow[d, "\mathcal{I}t"'] & CH_{S^n\vee S^n}\big(C^*(X)\big) \arrow[r, "\nabla_{X*}"] \arrow[d, "\mathcal{I}t"'] & CH_{S^n}\big(C^*(X)\big)^{\otimes 2}[-\dim(X)] \arrow[d, "\mathcal{I}t^{\otimes 2}"] \\
C^*\big(X^{S^n}\big) \arrow[r, "\delta_{S^n}^*"] & C^*\big(X^{S^n\vee S^n}\big) \arrow[r, "id\otimes\nabla_X"] & C^*\big(X^{S^n}\big)^{\otimes 2}[-\dim(X)]
\end{tikzcd}$$

is commutative in k-Mod_∞ (here the map ∇_{X*} is the map (56)*).*

Proof. This is a consequence of the naturality of the map $\mathcal{I}t : CH_X(C^*(Y)) \to C^*(Y^X)$, see Corollary 4.6. □

Proposition 8.15. *Let M be an n-connected oriented closed manifold whose homology groups are projective k-modules. Then the following diagram*

$$\begin{tikzcd}
CH^{S^n}\left(C^*(X), C^*(X)\right)^{\otimes 2} \arrow[r, "\cup_{S^n}"] \arrow[d, "\cong"'] & CH^{S^n}\left(C^*(X), C^*(X)\right) \arrow[d, "\cong"] \\
\left(C^*\big(X^{S^n}\big)[\dim(X)]\right)^{\otimes 2} \arrow[r, "\star_{S^n}"] & C^*\big(X^{S^n}\big)[\dim(X)]
\end{tikzcd}$$

is commutative in k-Mod_∞. Here the horizontal arrows are the sphere cup-product of Corollary 5.6 and the sphere product (55)*; the vertical arrows are given by the equivalences* (51) *(induced by the Poincaré duality map and Corollary 4.7).*

Since the vertical arrows are the maps defining the E_{n+1}-structure given by Theorem 8.1 on $C^*\big(X^{S^n}\big)[\dim(X)]$; it follows that the underlying commutative algebra structure on homology agrees with the sphere product.

Proof. Recall that there is a canonical isomorphism $CH^{S^n}(A, A^\vee) \cong \Big(CH_{S^n}(A, A^\vee)\Big)^\vee$. By our assumption on M, the canonical biduality map $C_*(X) \to \big(C^*(X)\big)^\vee$ is a quasi-isomorphism and it is sufficient to prove that the dual of the diagram depicted in Proposition 8.15 is commutative.

By definition (58) of μ_{S^n}, lemma 8.14 and lemma 8.13 we are left to prove that the map $id \otimes \nabla_M : C^*\big(M^{S^n\vee S^n}\big) \to C^*\big(M^{S^n}\big)^{\otimes 2}[-\dim(M)]$ sits inside a commutative diagram

(61)

$$\begin{tikzcd}
C^*\big(M^{S^n\vee S^n}\big) \arrow[r, "\pi^*"] \arrow[ddrr, "id\otimes\nabla_M"'] & C^*\big(\big(M^{S^n\vee S^n}\big)^{-TM}\big) \arrow[r, "\rho^\vee_{ev^*(th(M))}"] & C^*\big(\big(M^{S^n\vee S^n}\big)^{-TM}\big)[-\dim(M)] \arrow[d, "\text{collapse}^*"] \\
 & & C^*\big(M^{S^n\amalg S^n}\big)[-\dim(M)] \arrow[d, "\cong"] \\
 & & C^*\big(M^{S^n}\big)^{\otimes 2}[-\dim(M)]
\end{tikzcd}$$

in k-Mod$_\infty$.

Recall the equivalence (60) above. Under this equivalence, the cup-product by the pullback $ev^*(t(M))$ is given by

$$
\begin{aligned}
C^*\big(M^{S^n\vee S^n}\big) \cong C^*\big(M^{S^n\coprod S^n}\big) &\overset{\mathbb{L}}{\underset{C^*(M\times M)}{\otimes}} C^*(M) \\
\overset{id\otimes^{\mathbb{L}}_{C^*(M\times M)}\cup t(M)}{\longrightarrow} \quad C^*\big(M^{S^n\coprod S^n}\big) &\overset{\mathbb{L}}{\underset{C^*(M\times M)}{\otimes}} C^*(M)[-\dim(M)] \\
&\cong C^*\big(M^{S^n\vee S^n}\big)[-\dim(M)].
\end{aligned}
$$

Now, the commutativity of diagram (61) follows from Lemma 8.12. $\square$

9. Iterated Bar constructions

9.1. **Iterated loop spaces and iterated Bar constructions.** In this section we study the case of spaces of pointed maps from spheres to X, *i.e.* iterated loop spaces. The idea is to apply the formalism of higher Hochschild functor to the Bar construction of augmented E_∞-algebras.

Let (A,d) be a differential graded unital associative algebra (DGA for short) which is equipped with an augmentation $\epsilon: A\to k$. Denote $\overline{A} = ker(A\stackrel{\epsilon}{\to} k)$ the augmentation ideal of A. The standard bar construction on A is the chain complex $(Bar^{std}(A), b)$ defined by

$$Bar^{std}(A) = \bigoplus_{n\geq 1} \overline{A}^{\otimes n}$$

with differential given by

$$
\begin{aligned}
b(a_1\otimes\cdots a_n) &= \sum_{i=1}^{n} \pm a_1\otimes\cdots\otimes d(a_i)\otimes\cdots\otimes a_n \\
&\quad + \sum_{i=1}^{n-1} \pm a_1\otimes\cdots\otimes (a_i\cdot a_{i+1})\otimes\cdots\otimes a_n
\end{aligned}
$$

see [FHT, Fre2, KM] for details (and signs). Further, if A is a commutative differential graded algebra (CDGA for short), then the shuffle product makes the Bar construction $Bar^{std}(A)$ a CDGA as well.

Remark 9.1. Note that, by our convention on $\otimes_k$, if A is not flat over k, we replace it by a flat resolution. In particular $Bar^{std}: E_1 - Alg \to k\text{-Mod}_\infty$ preserves weak equivalences. This definition of $Bar^{std}(A)$ thus agrees with the classical one as soon as the underlying chain complex of A is flat over k.

Remark 9.2. The standard bar construction above extends naturally to A_∞-algebras. It also extends to any augmented E_1-algebra. Indeed, one can prove a Lemma similar to Lemma 9.3 below with factorization homology $\int_I(A,k)$ (of the E_1-algebra) instead of Hochschild chains over I (see [F1, F2, L-HA, AFT]). Note that, there is an natural equivalence $\int_I(A,k) \cong k\otimes^{\mathbb{L}}_A k$. In particular, if B is any DGA equivalent to A, then the standard bar construction of A is naturally equivalent to $Bar^{std}(B)$ in k-Mod$_\infty$.

Let A be an E_∞-algebra and let $\epsilon : A \to k$ be an augmentation. In particular, we can see k as an A-module thanks to the augmentation ϵ. In particular A is an E_1-algebra so that we can choose a DGA B and a quasi-isomorphism $f : B \to A$ of E_1-algebras. Then we define $Bar^{std}(A) := Bar^{std}(B)$. The fact that this construction is well-defined[26] indeed follows from the following lemma:

Lemma 9.3. *Let $I = [0,1]$ be the closed interval. There is a natural equivalence (in k-Mod_∞)*

$$CH_I(A) \underset{CH_{S^0}(A)}{\overset{\mathbb{L}}{\otimes}} k \;\cong\; Bar^{std}(A) \tag{62}$$

where $Bar^{std}(A)$ is the standard Bar construction. Further if A is a CDGA, the equivalence (62) is an equivalence of E_∞-algebras (where $Bar^{std}(A)$ is endowed with its CDGA-structure induced by the shuffle product).

Proof. Let $I_\bullet^{std}$ be the standard simplicial set model of the interval (viewed as a CW-complex with two vertices and one non-degenerate 1-cell). More precisely, $I_k^{std} = \{0, \dots, k+1\}$ with face maps d_i given, for $i = 0, \dots, k$, by $d_i(j)$ equal to j or $j-1$ depending on $j \le i$ or $j > i$. For any differential graded associative algebra B, one can form the simplicial dg-algebra

$$\widetilde{C_{I_\bullet^{std}}}(B) := (B^{\otimes I_k^{std}})_{k\geq 0} = (B \otimes B^k \otimes B)_{k\geq 0}$$

where the simplicial structure is defined as for Hochschild chains of a CDGA (it is immediate to check, and well known, that the commutativity is not necessary to check the simplicial identities in that case). Further the associated[27] differential graded module $DK(\widetilde{C_{I_\bullet^{std}}}(B))$ is the two-sided Bar construction $Bar^{std}(B,B,B)$ of B (see [GTZ, Example 2.3.4]). In particular, if $f : B \overset{\simeq}{\to} A$ is an equivalence of E_1-algebras, with B a DGA, then $DK(\widetilde{C_{I_\bullet^{std}}}(B)) \overset{f_*}{\to} Bar^{std}(A)$ is an equivalence (natural in A, B).

Now, forgetting the E_∞-structure of the Hochschild chain complex $CH_{I_\bullet^{std}}(A)$, we get a quasi-isomorphism of simplicial chain complexes

$$f : \widetilde{C_{I_\bullet^{std}}}(B) \xrightarrow{\simeq} \widetilde{CH_{I_\bullet^{std}}}(A)$$

and thus after taking the Dold-Kan ∞-functor $DK : sk\text{-}Mod_\infty \to k\text{-}\mathrm{Mod}_\infty$, we see that $CH_{I_\bullet^{std}}(A) \cong DK(\widetilde{C_{I_\bullet^{std}}}(B))$. Now the result follows since $Bar^{std}(B) \cong\cong \bigoplus_{n\geq 1} \overline{B}^{\otimes n}$ is the normalized chain complex associated to the simplicial chain complex $\widetilde{C_{I_\bullet^{std}}}(B)$, thus is quasi-isomorphic to $DK(\widetilde{C_{I_\bullet^{std}}}(B))$.

When A is a CDGA, the result follows from Corollary 3.7 and [GTZ, Section 2]. $\square$

In particular, we get an E_∞-lifting of the Bar construction of an E_∞-algebra that we denote

$$Bar(A) := CH_I(A) \underset{CH_{S^0}(A)}{\overset{\mathbb{L}}{\otimes}} k. \tag{63}$$

[26] i.e. independent of the choice of B

[27] via the usual Dold-Kan construction

Note that the augmentation $\epsilon : A \to k$ induces augmentations $\epsilon_* : CH_I(A) \to CH_I(k) \cong k$, $\epsilon_* : CH_{S^0}(A) \to CH_{S^0}(k) \cong k$ and thus an augmentation $Bar(A) \to k$ as well.

Since $Bar(A)$ is an augmented E_∞-algebra, we can take its Bar construction again.

Definition 9.4. The n^{th}-iterated Bar construction of an augmented E_∞-algebra A is the E_∞-algebra $Bar^{(n)}(A) = Bar(\cdots(Bar(A))\cdots)$.

Summing up the above results we have:

Proposition 9.5. *The n^{th}-iterated Bar construction Bar is an ∞-functor*

$$Bar^{(n)} : E_\infty\text{-}Alg \to E_\infty\text{-}Alg.$$

Further, there is an natural equivalence in E_∞-Alg between $B^{(n)}(A)$ and the n^{th}-iterated Bar construction defined by B. Fresse [Fre2].

Proof. Since $A \mapsto CH_I(A)$ is an ∞-endofunctor of E_∞-Alg, the same follows for Bar (and its iteration). By Lemma 9.3, the $Bar(A)$ is equivalent (in k-Mod$_\infty$) to $Bar^{std}(A)$ and, further, this equivalence is an equivalence in E_∞-Alg if A is a CDGA and $Bar^{std}(A)$ is endowed with the CDGA structure given by the shuffle product. Thus the uniqueness of the Bar construction in E_∞-Alg obtained in [Fre2] shows that $Bar^{(n)}$ is the correct n^{th}-iterated Bar construction. $\square$

Remark 9.6. Since the canonical map $CH_I(A) \to CH_{pt}(A) \cong A$ is an equivalence, we recover immediately from the excision axiom

$$Bar(A) \cong A \overset{\mathbb{L}}{\underset{CH_{S^0}(A)}{\otimes}} k \cong k \overset{\mathbb{L}}{\underset{A}{\otimes}} k.$$

Remark 9.7. In terms of factorization algebras, one has the following definition. Considered the unit interval with two stratified points given by its endpoints. Then, the analogue of Proposition 2.20 in that case is that locally constant (stratified) factorization algebra on I are the same as the data of an E_1-algebra A and a pair of left A-module M and a right A-module N. In particular taking the factorization algebra $\mathcal{A}$ for which A is augmented and $M = N = k$, we obtain that the factorization homology $\int_I \mathcal{A}$ (denoted $\int_I(A,k)$ in [F1]) is equivalent to the Bar construction, see [F1] for details.

There is an easy interpretation of the iterated Bar construction in terms of higher Hochschild chains. Note that, since k is an A-algebra (via the augmentation), $CH_{S^n}(A,k) \cong CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k$ is an E_∞-algebra.

Lemma 9.8. *There are natural equivalences of E_∞-algebras*

$$CH_{S^n}(A,k) \cong Bar^{(n)}(A).$$

Proof. Since $S^n \cong D^n \cup^h_{S^{n-1}} pt$, the homotopy invariance and excision axiom for Hochschild chains implies the following sequence of natural (in A) equivalences of E_∞-algebras

$$CH_{S^n}(A,k) \;\cong\; CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \;\cong\; CH_{I^n}(A) \overset{\mathbb{L}}{\underset{CH_{S^{n-1}}(A)}{\otimes}} k$$

Thus, for $n = 1$, the Lemma is proved (by Definition (63)).

Since $CH_X(k) \cong k$ for all $X \in Top_\infty$, by Corollary 3.27.(3), there are equivalences of E_∞-algebras

$$\begin{aligned} CH_I\Big(CH_{S^{n-1}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k\Big) &\cong CH_I\big(CH_{S^{n-1}}(A)\big) \overset{\mathbb{L}}{\underset{CH_I(A)}{\otimes}} k \\ &\cong CH_{(I\times S^{n-1})/I\times\{1\}}(A). \end{aligned}$$

where the last equivalence follows from Corollary 3.27.(4) and the excision axiom. Tensoring the last equivalence by $\overset{\mathbb{L}}{\underset{CH_{S^0}\left(CH_{S^{n-1}}(A,k)\right)}{\otimes}} k$ and applying the excision axiom again, we get

$$CH_I\Big(CH_{S^{n-1}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k\Big) \overset{\mathbb{L}}{\underset{CH_{S^0}\left(CH_{S^{n-1}}(A,k)\right)}{\otimes}} k \;\cong\; CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k.$$

Since the left hand side is $Bar\Big(CH_{S^{n-1}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k\Big)$, the Lemma now follows by induction. $\square$

We now study the coalgebra structure carried by the iterated Bar construction. Recall that the standard Bar construction of a DGA carries an natural associative coalgebra structure. We wish to apply the results of Section 5.2 to study the same result for E_n-coalgebras structures.

Recall the continuous map (19) $pinch : \mathcal{C}_n(r) \times S^n \longrightarrow \bigvee_{i=1\dots r} S^n$. Similarly to the definition of the map (20), applying the singular set functor to the map (19) we get a morphism

$$\begin{aligned} (64)\quad pinch_*^{S^n,r} : C_*\big(\mathcal{C}_n(r)\big) \otimes CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k & \\ \overset{pinch_*\otimes^{\mathbb{L}}_A id}{\longrightarrow} CH_{\bigvee_{i=1}^r S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \cong \Big(CH_{\coprod_{i=1}^r S^n}(A) \overset{\mathbb{L}}{\underset{A^{\otimes r}}{\otimes}} A\Big) \overset{\mathbb{L}}{\underset{A}{\otimes}} k & \\ \cong \Big(CH_{\coprod_{i=1}^r S^n}(A)\Big) \overset{\mathbb{L}}{\underset{A^{\otimes r}}{\otimes}} k \cong \Big(CH_{S^n}(A,k)\Big)^{\otimes r} & \end{aligned}$$

where the last equivalences follows from the excision axiom, the coproduct axiom and the definition of $CH_{S^n}(A,k)$.

Note that there is a canonical equivalence

$$(65)\qquad Hom_k\Big(CH_{S^n}(A) \otimes^{\mathbb{L}}_A k, k\Big) \cong RHom_A\Big(CH_{S^n}(A), k\Big) \cong CH^{S^n}(A,k).$$

Under this identification, the dual of the map (64) is the pinching map (20) from Section 5.1.

Proposition 9.9. *Let A be an E_∞-algebra and $\epsilon : A \to k$ an augmentation.*

(1) *The maps (64) $pinch_*^{S^n,r} : C_*\big(\mathcal{C}_n(r)\big) \otimes CH_{S^n}(A,k) \to \Big(CH_{S^n}(A,k)\Big)^{\otimes r}$ makes the iterated Bar construction $Bar^{(n)}(A) \cong CH_{S^n}(A,k)$ a natural E_n-coalgebra (in the $(\infty,1)$-category of E_∞-algebras)*

(2) *The dual E_n-algebra $RHom(Bar^{(n)}(A), k)$ is naturally equivalent to $CH^{S^n}(A,k)$ in E_n-Alg and thus to the centralizer $\mathfrak{z}(\epsilon)$ of the augmentation (viewed as a map of E_n-algebra by restriction).*

Proof. The proof of the first statement is similar to the proof of Theorem 5.11 (except that we take the predual of it). Fixing $c \in C_*(\mathcal{C}_n(r))$, all maps involved in the composition (64) defining $pinch_*^{S^n,r}(c,-)$ are maps of E_∞-algebras. Hence the structure maps of the E_n-coalgebra structures are compatible with the E_∞-structure.

Further, since the linear dual of the map (64) is the pinching map (20), statement (2) follows from Theorem 5.11, the equivalence

$$RHom(Bar^{(n)}(A),k) \cong RHom_A\Big(CH_{S^n}(A),k\Big) \cong CH^{S^n}(A,k)$$

and Corollary 7.23 $\square$

If Y is a pointed space, its E_∞-algebra of cochains $C^*(Y)$ has a canonical augmentation $C^*(Y) \to C^*(pt) \cong k$ induced by the base point $pt \to Y$. Tensoring the map $\mathcal{I}t : CH_{S^n}(C^*(Y)) \longrightarrow C^*\big(Y^{S^n}\big)$ (given by Theorem 4.4) with $\otimes^{\mathbb{L}}_{C^*(Y)} k$ yields a natural E_∞-algebra morphism

$$(66)\quad \mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \cong CH_{S^n}(C^*(Y),k) \stackrel{\mathcal{I}t\otimes^{\mathbb{L}}_{C^*(Y)}k}{\longrightarrow} C^*\big(Y^{S^n}\big) \otimes^{\mathbb{L}}_{C^*(Y)} k \longrightarrow C^*\big(\Omega^n(Y)\big)$$

where the last map is induced by applying the singular cochain functor to $\Omega^n(Y) \cong Y^{S^n} \times^h_Y pt$.

Further, using the equivalence (65), the linear dual of this map (composed with the canonical biduality morphism) yields a map

$$(67)\quad \mathcal{I}t_{\Omega^n} : C_*\big(\Omega^n(Y)\big) \longrightarrow C^*\big(\Omega^n(Y)\big)^\vee \longrightarrow CH^{S^n}(C^*(Y),k) \cong \Big(Bar^{(n)}(C^*(Y))\Big)^\vee$$

We can now state our main application to iterated loop spaces, generalizing classical results in algebraic topology. Since the iterated loop space $\Omega^n(Y)$ are E_n-algebras in spaces, $C^*(\Omega^n(Y))$ is an $E_n - coalgebra$ in E_∞-Alg and $C_*(\Omega^n(Y))$ an E_n-algebra (in E_∞-coAlg, the $(\infty,1)$-category of E_∞-coalgebras).

Corollary 9.10. *Let Y be a pointed topological space.*

(1) *The map* (66) $\mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \to C^*\big(\Omega^n(Y)\big)$ *is an E_n-coalgebra morphism in the category of E_∞-algebras. It is further an equivalence if Y is n-connected.*
(2) *Dually, the map* (67) $\mathcal{I}t_{\Omega^n} : C_*\big(\Omega^n(Y)\big) \longrightarrow \Big(Bar^{(n)}(C^*(Y))\Big)^\vee$ *is an E_n-algebra morphism (in k-Mod$_\infty$). Further, if k is a field, Y is n-connected and has finite dimensional homology groups, then $\Big(Bar^{(n)}(C^*(Y))\Big)^\vee$ is an E_∞-coalgebra and the map* (67) $\mathcal{I}t_{\Omega^n}$ *is an equivalence of E_n-algebras in E_∞-coAlg.*

In particular, the Hochschild chains over the spheres is a model for the natural E_n-algebra structure on $C_*(\Omega^n Y)$.

Proof. By Theorem 4.4, the map $\mathcal{I}t : CH_{S^n}(C^*(Y)) \longrightarrow C^*\big(Y^{S^n}\big)$ is an E_∞-algebra map and thus so is $\mathcal{I}t_{\Omega^n}$. Further, Theorem 4.4 gives an natural transformation

$$\mathcal{I}t : CH^*_X\big(C^*(Y)\big) \longrightarrow C^*\big(Y^X\big)$$

from which we deduce a commutative diagram

$$\begin{tikzcd} CH_{I^n}\big(C^*(Y)\big) \underset{CH_{S^{n-1}}(C^*(Y))}{\overset{\mathbb{L}}{\otimes}} C^*(Y) \underset{C^*(Y)}{\overset{\mathbb{L}}{\otimes}} k \arrow[r, "\simeq"] \arrow[d, "\mathcal{I}t\otimes^{\mathbb{L}}_{\mathcal{I}t} id"'] & CH_{S^n}\big(C^*(Y)\big) \underset{C^*(Y)}{\overset{\mathbb{L}}{\otimes}} k \arrow[d, "\mathcal{I}t\otimes^{\mathbb{L}}_{C^*(Y)} id"] \\ C^*\big(Y^{I^n}\big) \underset{C^*(Y^{S^{n-1}})}{\overset{\mathbb{L}}{\otimes}} C^*(Y) \underset{C^*(Y)}{\overset{\mathbb{L}}{\otimes}} k \arrow[r] & C^*\big(Y^{S^n}\big) \underset{C^*(Y)}{\overset{\mathbb{L}}{\otimes}} k \end{tikzcd} \tag{68}$$

in E_∞-Alg in which the horizontal arrows are induced by the homotopy pushout $\Omega^n X \cong X^{I^n} \cup_{X^{S^{n-1}}} pt$. The lower horizontal arrow is an equivalence when Y is n-connected. Further, the map $\mathcal{I}t : CH_{S^{n-1}}\big(C^*(Y)\big) \to C^*\big(Y^{S^{n-1}}\big)$ is an equivalence when Y is $n-1$-connected by Theorem 4.4. Since the map induced by the base point $C^*(Y) \to CH_{I^n}(C^*(Y))$ is an equivalence, the map $\mathcal{I}t : CH_{I^n}(C^*(Y)) \to C^*\big(Y^{I^n}\big)$ is an equivalence when Y is connected. Thus, we deduce from the commutativity of diagram (68) that the map $\mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \to C^*\big(\Omega^n(Y)\big)$ is an equivalence when Y is n-connected.

In order to finish the proof of Assertion 1 in Corollary 9.10, it remains to check that $\mathcal{I}t^{\Omega^n}$ is a map of E_n-coalgebras. By definition, the E_n-coalgebra structure of $C^*\big(\Omega^n(Y)\big)$ is induced by taking the singular cochains functor (from Top_∞ to E_∞-Alg) to the E_n-algebra structure of $\Omega^n(Y)$ which is the (homotopy pullback) $\Omega^n(Y) \cong \big(Y^{S^n} \times_Y pt\big)$. By definition the E_n-algebra structure of $\Omega^n(Y)$ is induced by the pinching map (20) $\mathcal{C}_n(r) \times S^n \to \bigvee_{i=1\dots r} S^n$. Indeed, since the pinching map preserves the base point of S^n, we have the following composition

$$\begin{aligned} \mathcal{C}_n(r) \times \big(Y^{S^n} \times^h_Y pt\big)^r &\xrightarrow{\cong} \mathcal{C}_n(r) \times \big(Y^{\coprod_{i=1\dots r} S^n}\big) \times_{Y^r} pt \\ &\xrightarrow{\cong} \big(Y^{\bigvee_{i=1\dots r} S^n}\big) \times_Y pt \xrightarrow{pinch^*} Y^{S^n} \times_Y pt. \end{aligned} \tag{69}$$

By naturality of $\mathcal{I}t$, we have a commutative diagram

$$\begin{tikzcd} CH_{S^n}\big(C^*(Y)\big) \underset{C^*(Y)}{\overset{\mathbb{L}}{\otimes}} k \arrow[r, "pinch_* \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} id"] \arrow[d, "\mathcal{I}t\otimes^{\mathbb{L}}_{C^*(Y)} id"'] & CH_{\bigvee_{i=1\dots r} S^n}\big(C^*(Y)\big) \underset{C^*(Y)}{\overset{\mathbb{L}}{\otimes}} k \arrow[d, "\mathcal{I}t\otimes^{\mathbb{L}}_{C^*(Y)} id"] \\ C^*\big(Y^{S^n}\big) \underset{C^*(Y)}{\overset{\mathbb{L}}{\otimes}} k \arrow[r, "C^*(pinch^*)"] & C^*\big(Y^{\bigvee_{i=1\dots r} S^n}\big) \underset{C^*(Y)}{\overset{\mathbb{L}}{\otimes}} k \end{tikzcd} .$$

The commutativity of this diagram, together with the definition of the map (64) $pinch_*^{S^n,r} : C_*\big(\mathcal{C}_n(r)\big) \otimes CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \to \Big(CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k\Big)^{\otimes r}$ giving the E_n-coalgebra structure of $Bar^{(n)}(C^*(Y))$, and the fact that the E_n-coalgeba structure of $C^*(\Omega^n(Y))$ is given by applying the functor $C^*(-)$ to the composition (69) show that $\mathcal{I}t^{\Omega^n}$ is an E_n-algebra map.

The proof of the fact that $\mathcal{I}t_{\Omega^n}$ is a map of E_n-algebra is similar, using in addition the naturality of the biduality morphism $C \to C^{\vee\vee}$ and Corollary 4.7. Further, when k is a field and the groups $H_\ell(Y)$ are finitely generated, then $C_*(Y) \to (C^*(Y))^\vee$ is an equivalence. Further, if Y is n-connected, it follows

from the Eilenberg-Moore spectral sequence that $Bar^{(n)}(C^*(Y))$ has finite dimensional homology groups. Hence, the dual $\left(Bar^{(n)}(C^*(Y))\right)^\vee$ inherits an natural E_∞-coalgebra structure (dual of the E_∞-algebra structure of $Bar^{(n)}(C^*(Y))$). It is then immediate to check that the arguments to prove Statement (1) above can be dualized to prove that $\mathcal{I}t_{\Omega^n}$ is also an equivalence of E_∞-coalgebras. $\square$

Remark 9.11. A careful analysis of the proof of Corollary 9.10 shows that the assumption that Y is n-connected can be replaced by the assumption that the cohomological Eilenberg-Moore spectral sequence of the path space fibration is strongly convergent for all $\Omega^i Y$ ($i \le n$).

Remark 9.12. Statement (2) in Corollary 9.10 is somehow unsatisfying since one recovers an E_∞-coalgebra structure on right hand side $\left(Bar^{(n)}(C^*(Y))\right)^\vee$ only when the biduality morphism $Bar^{(n)}(C^*(Y)) \to \left(Bar^{(n)}(C^*(Y))\right)^{\vee\vee}$ is an equivalence (while the left hand side has always such a structure). The reason for it, is that this statement is in fact the bidual of a statement involving iterated *coBar* construction of E_∞-coalgebras.

Indeed, one can define Hochschild cochains over spaces for E_∞-coalgebras in a similar way to what we do in Section 3 getting an ∞-functor $CH : Top^{op}_\infty \times E_\infty\text{-}coAlg \to E_\infty\text{-}coAlg$ ($(X,C) \mapsto CH^X(C)$). For instance, one has an natural equivalence $CH^X(C) \cong C \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^{\otimes}}{\otimes}} C^*(X)$ similar to Proposition 3.6.

All results of Section 3, Section 4 and Section 5 have "dual" counterparts which can be proved similarly. We claim that there is an iterated cobar construction $coBar^{(n)} : E_\infty\text{-}coAlg \to E_n\text{-}Alg(E_\infty\text{-}coAlg)$ defined similarly to this Section 9.1 and that further there is an natural E_n-algebra map $coBar^{(n)}(C_*(Y)) \to C_*(\Omega^n(Y))$ in E_∞-coAlg which is an equivalence when Y is n-connected. We leave the many details to fill to the interested reader.

9.2. **Iterated Bar constructions of augmented E_n-algebras.** In this section we explain how to generalize the iterated Bar construction for E_∞-algebras in § 9.1 to E_n-algebras. In particular we describe the E_n-coalgebra structure of the n-times iterated Bar construction. Our definition and study of the Bar construction follows the ones given by Francis [F1] and Lurie [L-HA].

In this section we assume A is an augmented E_n-algebra and we denote $\epsilon : A \to k$ the augmentation (which is a map of E_n-algebras). In particular, we endow k with its structure of A-E_n-module given by the augmentation. We denote $E_n\text{-}Alg^{aug}$ *the* $(\infty,1)$*-category of augmented* E_n*-algebras.*

For an augmented E_n-algebra, Definition (63) and Lemma 9.3 suggest to define

$$Bar(A) := \int_{I\times\mathbb{R}^{n-1}} A \overset{\mathbb{L}}{\underset{\int_{S^0\times\mathbb{R}^{n-1}} A}{\otimes}} k \tag{70}$$

where $k \cong \int_{I\times\mathbb{R}^{n-1}} k$ is endowed its natural structure of A-E_1-module. This definition agrees with the usual one:

Lemma 9.13 (Francis [F1])**.** *There is a natural equivalence (in k-Mod_∞)*

$$Bar(A) \;\cong\; Bar^{std}(A) \;\cong\; k \otimes^{\mathbb{L}}_A k \tag{71}$$

where $Bar^{std}(A)$ is the standard Bar construction as in § 9.1.

When X be a manifold of dimension d equipped with a framing of $X \times \mathbb{R}^k$, then for any E_{d+k}-algebra B, $\int_{X\times\mathbb{R}^k} B$ is canonically an E_k-algebra, see [L-HA, F1] for details. Note that this follows from Theorem 2.14 and the fact that a factorization algebra on $X\times\mathbb{R}^k$ are the same as factorization algebras on X with values in E_d-Alg see Proposition 2.15 (or [GTZ2]). Applying this to $X = I$ or $X = S^0$ we get the following result which is also asserted in [F1, L-HA].

Proposition 9.14. *The Bar construction* (71) *for augmented E_m-algebras ($m \geq 1$) has a canonical lift*

$$Bar : E_m\text{-}Alg^{aug} \to E_{m-1}\text{-}Alg^{aug}$$

which coincides for E_∞-algebras with the one given in § 9.1 and further sits into a commutative diagram

$$\begin{array}{ccccccccc}
E_1\text{-}Alg & \longleftarrow & E_2\text{-}Alg & \longleftarrow \cdots \longleftarrow & E_m\text{-}Alg & \longleftarrow \cdots \longleftarrow & E_\infty\text{-}Alg \\
\downarrow{\scriptstyle Bar} & & \downarrow{\scriptstyle Bar} & & \downarrow{\scriptstyle Bar} & & \downarrow{\scriptstyle Bar} \\
k\text{-}Mod_\infty & \longleftarrow & E_1\text{-}Alg & \longleftarrow \cdots \longleftarrow & E_{m-1}\text{-}Alg & \longleftarrow \cdots \longleftarrow & E_\infty\text{-}Alg
\end{array}$$

where the horizontal arrows are the canonical forget functors induced by the tower of maps of operads (2).

Proof. By the above and Theorem 2.14, we have that $\int_{S^0\times\mathbb{R}^{m-1}} A$, $\int_{I\times\mathbb{R}^{m-1}} A$ and k are (global sections of) locally constant factorizations algebras over $\mathbb{R}^{m-1}$; in particular, we can see $\int_{S^0\times\mathbb{R}^{m-1}} A$ as an E_1-algebra in the symmetric monoidal category of E_{m-1}-algebras, *i.e.*, locally constant factorizations algebras over $\mathbb{R}^{m-1}$. Similarly $\int_{I\times\mathbb{R}^{m-1}} A$ is a left module over $\int_{S^0\times\mathbb{R}^{m-1}} A$ in the symmetric monoidal category of E_{m-1}-algebras, *i.e.*, it belongs to $\big(\int_{S^0\times\mathbb{R}^{m-1}} A\big)\text{-}Mod^{E_1}\big(E_{m-1}\text{-}Alg\big)$. This shows that the Bar construction is an object of E_{m-1}-Alg. Further, the augmentation $\epsilon : A \to k$ induces a maps $\int_{I\times\mathbb{R}^{m-n}} \epsilon : \int_{I\times\mathbb{R}^{m-1}} A \to k$ which is a map of locally constant factorization algebras on $\mathbb{R}^{m-n}$ hence of E_{m-n}-algebras. Similarly $\int_{S^0\times\mathbb{R}^{m-1}} A \to k$ is a map of E_{m-n+1}-algebras; hence ϵ induces an augmentation $Bar(A) \to k$ in E_{m-n}-Alg. The equivalence of the two definitions for E_∞-algebras is an immediate consequence of Theorem 3.11 or [GTZ2, Theorem 5]. The commutativity of the diagram follows from the fact that $E_m\text{-}Alg \to E_{m-1}\text{-}Alg$ is induced by the map of ∞-operad $\mathbb{E}^{\otimes}_{m-1} \to \mathbb{E}^{\otimes}_m$ induced by taking the product of $m-1$-dimensional disks with the interval $\mathbb{R}$, *i.e*, it is induced by the pushforward of factorization algebras along the projection $\mathbb{R}^{m-1}\times\mathbb{R} \to \mathbb{R}^{m-1}$. $\square$

By Proposition 9.14, we can iterate (up to m-times) the Bar constructions of an E_m-algebra.

Definition 9.15. Let $0 \leq n \leq m$. The n^{th}-iterated Bar construction of an augmented E_m-algebra A is the E_{m-n}-algebra (given by Proposition 9.14)

$$Bar^{(n)}(A) := Bar(\cdots(Bar(A))\cdots).$$

Proposition 9.14 implies that Definition 9.15 agrees with Definition 9.4 for E_∞-algebras.

Remark 9.16. The iterated Bar construction given in Definition 9.15 should be closely related to the one (obtained at the level of model categories) by Fresse [Fre3].

The following result, due to Francis [F1, Lemma 2.44], identifies the iterated Bar construction in terms of factorization homology

Lemma 9.17 (Francis)**.** *Let A be an E_m-algebra and $0 \le n \le m$. There is an natural equivalence of E_{m-n}-algebras*

$$Bar^{(n)}(A) \cong \int_{I^m \times \mathbb{R}^{m-n}} A \overset{\mathbb{L}}{\underset{\int_{S^{m-1} \times \mathbb{R}^{m-n+1}} A}{\otimes}} k$$

Proof. This is essentially Lemma 2.44 together with Corollary 3.32 in [F1]. Alternatively, on can use a proof similar to the one of Lemma 9.8 replacing $CH_{I^n}(A)$ with the E_{m-n}-algebra $\int_{I^m \times \mathbb{R}^{m-n}} A$ using excision for factorization homology (see [F1, AFT, GTZ2]), and the Fubini theorem for factorization homology [GTZ2, Corollary 17] instead of Corollary 3.27.(4). $\square$

Remark 9.18. Let $\epsilon : A \to k$ be an augmented E_m-algebra. Identify I^n with the closed unit disk in $\mathbb{R}^n$ and let $D^n = I^n \setminus \partial I_n$ be its interior. Then we can define a stratified locally constant factorization algebra on $I^n \times \mathbb{R}^{m-n}$, denoted (A, k), on I^n which associates the E_m-algebra A on any disk inside $D^n \times \mathbb{R}^{m-n}$ and takes the value k on any disk in an neighborhood of $(\partial I^n) \times \mathbb{R}^{m-n}$; the factorization algebra structure being induced by the A-E_n-module structure of k. Then, there is an natural equivalence of E_{m-n}-algebras (see [F1, AFT] or apply the excision property for factorization homology)

$$Bar^{(n)}(A) \ \cong \ \int_{I^n \times \mathbb{R}^{m-n}} (A, k).$$

We now describe an E_n-coalgebra structure on $Bar^{(n)}(A)$ by defining it as a locally constant $N(\mathrm{Disk}(\mathbb{R}^n)$-coalgebra. We may assume A is a locally constant factorization algebra on $\mathbb{R}^m$, denoted $\mathcal{A}$ (in particular $\mathcal{A}(D) \cong \int_D A$ for any disk $D \subset \mathbb{R}^m$).

Let $\phi : \mathbb{R}^n \overset{\simeq}{\rightarrow} U \subset \mathbb{R}^n$ be an embedding of a disk in $\mathbb{R}^n$. Write $\partial U := U \setminus \{\phi(I^n)\}$ We associate to this the E_{n-m}-algebra

$$\begin{aligned} Bar^{(n)}(A)(\phi) &:= \int_{I^n \times \mathbb{R}^{m-n}} \mathcal{A}(U \times \mathbb{R}^{m-n}) \overset{\mathbb{L}}{\underset{\int_{S^{m-1} \times \mathbb{R}^{m-n+1}} \mathcal{A}(U \times \mathbb{R}^{m-n})}{\otimes}} k \\ &\cong Bar^{(n)}\big(\mathcal{A}(U \times \mathbb{R}^{m-n})\big). \end{aligned} \tag{72}$$

Now let $\phi_i : \mathbb{R}^n \overset{\simeq}{\rightarrow} U_i \subset \mathbb{R}^n$ $(i = 1 \dots r)$ be a family of embeddings with pairwise disjoint images and let $h : \coprod_{i=1}^r \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to V \subset \mathbb{R}^n$ be embeddings such that the following diagram

$$\begin{tikzcd} \coprod_{i=1}^r \mathbb{R}^n \arrow[rr, "h"] \arrow[dr, "\coprod \phi_i"'] & & \mathbb{R}^n \arrow[dl, "\psi"] \\ & V \subset \mathbb{R}^n & \end{tikzcd} \tag{73}$$

is commutative.

Remark 9.19 (*sketch of the construction*)**.** We start by sketching the construction of the E_n-coalgebra structure following the above Remark 9.18; we can think of the iterated Bar construction on V as a stratified factorization algebra $D \mapsto Bar^{(n)}(A)(D)$ on the closure $\overline{V}$ of V (which assigns the A-E_n-module k to balls

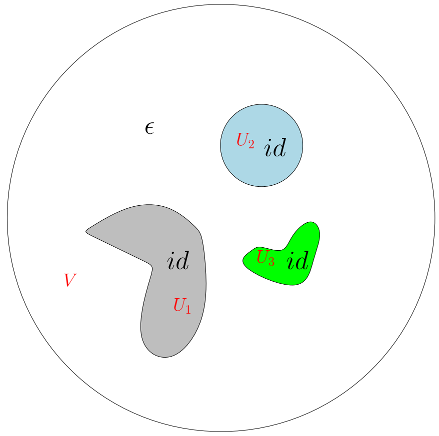

FIGURE 2. The map $Bar^{(n)}(A)(V) \to Bar^{(n)}(A)(U_1) \otimes Bar^{(n)}(A)(U_2) \otimes Bar^{(n)}(A)(U_3)$

in a neighborhood of the boundary $\overline{V} \setminus V$). The data of $\phi_1, \dots, \phi_r$ allow to define another stratified factorization algebra $\mathcal{F}$ where $\overline{V}$ is stratified with one open strata given by the union of the disks $\bigcup_{i=1}^r U_i$ and one closed strata given by their complement $\overline{V} \setminus \big(\bigcup_{i=1}^r U_i\big)$. Then $\mathcal{F}$ is defined as the rule which to each ball D inside U_i associates $\int_D A$, *i.e.*, A and which associates $\mathcal{F}(D) = k$ on the closed strata, *i.e.*, for any disk in a small neighborhood (meaning it should avoid all $\phi_i(0)$) of the closed strata. The factorization algebra structure is given by the A-E_n-module structure of k and is well defined since the balls D above defined are a basis of the topology of $\overline{V}$. Then, the global section $\mathcal{F}(\overline{V})$ is easily seen to be equivalent to $\big(Bar^{(n)}(A)\big)^{\otimes r}$. Further, the map which is the identity on each disk D inside a U_i and is the augmentation $\epsilon : A \to k$ on each disk in a small neighborhood of the closed strata $\overline{V} \setminus \big(\bigcup_{i=1}^r U_i\big)$ defines a map of factorization algebra $Bar^{(n)}(A)(W) \mapsto \mathcal{F}(W)$ and thus a structure map

$$Bar^{(n)}\big(\mathcal{A}(V \times \mathbb{R}^{m-n})\big) \longrightarrow Bar^{(n)}\big(\mathcal{A}(U_1 \times \mathbb{R}^{m-n})\big) \otimes \cdots \otimes Bar^{(n)}\big(\mathcal{A}(U_\times \mathbb{R}^{m-n})\big)$$

defining the desired E_n-coalgebra structure. See Figure 2 (in the case $r = 3$). We explain the construction in more details below.

To describe the E_n-coalgebra structure, we wish to define a map

$$\gamma_{U_1,\dots,U_r,V} : Bar^{(n)}\big(\mathcal{A}(V \times \mathbb{R}^{m-n})\big) \\ \longrightarrow Bar^{(n)}\big(\mathcal{A}(U_1 \times \mathbb{R}^{m-n})\big) \otimes \cdots \otimes Bar^{(n)}\big(\mathcal{A}(U_\times \mathbb{R}^{m-n} r)\big).$$

Since $\mathcal{A}_{|V \times \mathbb{R}^{m-n}}$ is a locally constant factorization algebra, its values are determined by its values on the sub-family $\mathcal{D}$ of disks consisting of those disks which are either totally included in a $U_i \times \mathbb{R}^{m-n}$ or in the complement $V \times \mathbb{R}^{m-n} \setminus \{\phi_i(I^n) \times \mathbb{R}^{m-n}, i = 1 \dots r\}$ of the $\phi_i(I^n) \times \mathbb{R}^{m-n}$ (clearly $\mathcal{D}$ is a basis of opens in V which is stable by intersection). Thus the map $\gamma_{U_1,\dots,U_r,V}$ is determined by its value on any tensor product : $\mathcal{A}(D_1) \otimes \cdots \otimes \mathcal{A}(D_\ell)$ where the D_j's are pairwise disjoint disks in $\mathcal{D}$. We can reorganize the D_j's into $r+1$ collections $(D_{1,j_1})_{j_1 \in J_1}$, ..., $(D_{r,j_r})_{j_r \in J_r}$, $(D_{\partial,j_\partial})_{j_\partial \in J_\partial}$ where the D_{i,j_i} are in $U_i \times \mathbb{R}^{m-n}$ and the D_{∂,j_∂} are in

$V \times \mathbb{R}^{m-n} \setminus \{\phi_i(I^n) \times \mathbb{R}^{m-n},\, i = 1 \dots r\}$. Then we define

$$\tilde{\gamma}_{U_1,\dots,U_r,V} : \Big(\bigotimes_{J_1} \mathcal{A}(D_{1,j_1})\Big) \otimes \cdots \otimes \Big(\bigotimes_{J_\partial} \mathcal{A}(D_{\partial,j_\partial})\Big) \stackrel{h_1\otimes\cdots\otimes h_r\otimes\epsilon}{\longrightarrow} Bar^{(n)}\big(\mathcal{A}(U_1 \times \mathbb{R}^{m-n})\big) \otimes \cdots \otimes Bar^{(n)}\big(\mathcal{A}(U_r \times \mathbb{R}^{m-n})\big)$$

where ϵ is the augmentation applied on each $\mathcal{A}(D_{\partial,j_\partial})$ and $h_i : \Big(\bigotimes_{J_1} \mathcal{A}(D_{i,j_i})\Big) \to Bar^{(n)}\big(\mathcal{A}(U_i \times \mathbb{R}^{m-n})\big)$ is the composition

$$h_i : \Big(\bigotimes_{J_1} \mathcal{A}(D_{i,j_i})\Big) \longrightarrow \mathcal{A}(U_i \times \mathbb{R}^{m-n}) \longrightarrow Bar^{(n)}\big(\mathcal{A}(U_i \times \mathbb{R}^{m-n})\big)$$

where the first map is given by the structure map of the factorization algebra $\mathcal{A}$ and the last one by tensoring $\mathcal{A}(U_i \times \mathbb{R}^{m-n})$ with k.

Lemma 9.20. *The map $\tilde{\gamma}_{U_1,\dots,U_r,V}$ is independent of the choices and passes to the derived tensor product to define a map of augmented E_{m-n}-algebras.*

$$\gamma_{u_1,\dots,U_r,V} : Bar^{(n)}\big(\mathcal{A}(V \times \mathbb{R}^{m-n})\big) \longrightarrow Bar^{(n)}\big(\mathcal{A}(U_1 \times \mathbb{R}^{m-n})\big) \otimes \cdots \otimes Bar^{(n)}\big(\mathcal{A}(U_r \times \mathbb{R}^{m-n})\big).$$

Proof. The only choices we made are when we reorganize the D_j's into the $r+1$-collections. Namely a disk D lying in the intersection of U_i and $V \times \mathbb{R}^{m-n} \setminus \{\phi_i(I^n) \times \mathbb{R}^{m-n},\, i = 1 \dots r\}$ can be seen assigned to $(D_{i,j_i})_{j_i \in J_i}$ or D_{∂,j_∂}. By definition of the module structure of $\mathcal{A}(U_i \times \mathbb{R}^n)$, $h_i : \mathcal{A}(D) \to Bar^{(n)}(\mathcal{A}(U_i \times \mathbb{R}^{m-n}))$ is the augmentation ϵ. It follows that the two possible ways of assigning a family to D yields the same value for the map $\tilde{\gamma}_{U_1,\dots,U_r,V}$. Furthermore, let us choose a cube C in $\mathbb{R}^n$ which contains the image $h(\coprod_{i=1}^r \mathbb{R}^n)$. Then,

$$Bar^{(n)}(\mathcal{A}(V \times \mathbb{R}^{m-n})) \cong \int_{I^n \times \mathbb{R}^{m-n}} \mathcal{A}(V \times \mathbb{R}^{m-n}) \overset{\mathbb{L}}{\underset{\int_{(V \setminus \psi(C)) \times \mathbb{R}^{m-n}} A}{\otimes}} k.$$

Since the map $\tilde{\gamma}_{U_1,\dots,U_r,V}$ acts by the augmentation on $\mathcal{A}(D)$ for any disk D in $(V \setminus \psi(C)) \times \mathbb{R}^{m-n}$, it then follows that $\tilde{\gamma}_{U_1,\dots,U_r,V}$ induces a map $Bar^{(n)}\big(\mathcal{A}(V \times \mathbb{R}^{m-n})\big) \to \bigotimes_{i=1}^r Bar^{(n)}\big(\mathcal{A}(U_i \times \mathbb{R}^{m-n})\big)$. The fact that $\gamma_{u_1,\dots,U_r,V}$ is a map of augmented E_{m-n}-algebras follows from the same argument used in Proposition 9.14 to prove that the Bar construction takes values in augmented E_{m-1}-algebras and the fact that the tensor product of augmented E_l-algebras has an natural augmentation given by the tensor product of augmentations. □

Proposition 9.21. *Let A be an augmented E_m-algebra and $0 \le n \le m$.*

(1) *The maps $\gamma_{u_1,\dots,U_r,V}$ makes $Bar^{(n)}(A)$ an E_n-coalgebra (induced by a locally constant $N(Disk)(\mathbb{R}^n)$-coalgebra structure in the sense of § 2.3) in the $(\infty,1)$-category of E_{m-n}-algebras, which for $m = \infty$ coincides with the construction of § 9.1. This construction is functorial,* i.e.*, the iterated Bar construction (Definition 9.15) lifts as a functor of $(\infty,1)$-categories*

$$Bar^{(n)} : E_m\text{-}Alg^{aug} \longrightarrow E_n\text{-}coAlg\Big(E_{m-n}\text{-}Alg^{aug}\Big).$$

(2) *The dual $RHom(Bar^{(n)}(A), k)$, endowed with the dual E_n-algebra structure in E_{m-n}-Alg induced by (1), is the centralizer $\mathfrak{z}(A \stackrel{\epsilon}{\to} k)$ of the augmentation (see § 7.3).*

(3) *$Bar^{(1)}(A)$ is equivalent as an E_1-coalgebra to the standard (§ 9.1) Bar construction $Bar^{std}(A)$ and $Bar^{(n)}(A)$ is equivalent to the iterated Bar constructions of* [F1] *(in the ∞-category E_n-$coAlg\Big(E_{m-n}$-$Alg^{aug}\Big)$).*

Proof. From § 2.3 and § 2.4 (in particular the coalgebra analogue of Proposition 2.15), we only need to prove that $Bar^{(n)}(A)$ is the global section $Bar^{(n)}(A)(\mathbb{R}^n)$ of a locally constant $N(\mathrm{Disk}(\mathbb{R}^n))$-coalgebra. As previously noticed in Remark 7.11, this is equivalent to proving that the maps $\gamma_{u_1,\dots,U_r,V}$ given by Lemma 9.20 are the structure maps

$$\gamma_{\phi_1,\dots,\phi_r,\psi} : Bar^{(n)}(A)(\psi) \longrightarrow Bar^{(n)}(A)(\phi_1) \otimes \cdots \otimes Bar^{(n)}(A)(\phi_r)$$

of a locally constant $N(\mathrm{Disk}(\mathbb{R}^n)')$-coalgebra; here the ϕ_i and ψ are the embeddings sitting in the commutative diagram (73). By Lemma 9.20, we already know that these maps are maps of (augmented) E_{m-n}-algebras. The naturality of the structure maps with respect to inclusions of disks, *i.e.*, the identity

$$\Big(\gamma_{\theta_1^1,\dots,\theta_{i_1}^1,\phi_1} \otimes \cdots \otimes \gamma_{\theta_1^r,\dots,\theta_{i_r}^r,\phi_r}\Big) \circ \gamma_{\phi_1,\dots,\phi_r,\psi} = \gamma_{\theta_1^1,\dots,\theta_{i_1}^1,\dots,\theta_1^r,\dots,\theta_{i_r}^r,\psi}.$$

for families of embeddings $\theta_i^j : \mathbb{R}^n \stackrel{\simeq}{\to} W_i^j \subset \mathbb{R}^n$ with pairwise disjoint images lying in the (pairwise disjoint) images of $\phi_j : \mathbb{R}^n \stackrel{\simeq}{\to} U_j \subset V \subset \mathbb{R}$ where $V = \psi(\mathbb{R}^n)$ as before (and the embeddings θ_i^j are related to ϕ_j by a diagram similar to (73)). The proof is the same as for the naturality of the structure maps $\rho_{U_1,\dots,U_r,V}$ in the proof of Theorem 7.7.

By assumption A is an E_m-algebra and $\mathcal{A}$ a locally constant factorization algebra on $\mathbb{R}^m$. Hence, for any inclusion $U \hookrightarrow V$ of two disks, the map $U \times \mathbb{R}^{m-n} \hookrightarrow V \times \mathbb{R}^{m-n}$ is an inclusion of a disk inside a bigger disk so that the canonical map $\mathcal{A}(U \times \mathbb{R}^{m-n}) \to \mathcal{A}(V \times \mathbb{R}^{m-n})$ is an equivalence of $\mathcal{A}(U \times \mathbb{R}^{m-n})$-$E_m$-modules. Further, the following diagram

$$\begin{tikzcd}
\mathcal{A}(V \times \mathbb{R}^{m-n}) \otimes^{\mathbb{L}}_{\mathcal{A}(U\times\mathbb{R}^{m-n})} k \arrow[r, "\tilde{\gamma}_{U,V}"] & \mathcal{A}(U \times \mathbb{R}^{m-n}) \otimes^{\mathbb{L}}_{\mathcal{A}(U\times\mathbb{R}^{m-n})} k \\
\mathcal{A}(U \times \mathbb{R}^{m-n}) \otimes^{\mathbb{L}}_{\mathcal{A}(U\times\mathbb{R}^{m-n})} k \arrow[u, "\simeq"] \arrow[ur, "id"'] &
\end{tikzcd}$$

is commutative since $\tilde{\gamma}_{U,V}$ restricted to open disks in $U \times \mathbb{R}^{m-n}$ is the identity functor. It follows that the induced map

$$Bar^{(n)}\big(\mathcal{A}(U \times \mathbb{R}^{m-n})\big) \stackrel{\gamma_{U,V}}{\longrightarrow} Bar^{(n)}\big(\mathcal{A}(V \times \mathbb{R}^{m-n})\big)$$

is an equivalence as well. Hence (see construction (72)), the rule $\phi \mapsto Bar^{(n)}(A)(\phi)$ is a locally constant $N(\mathrm{Disk}(\mathbb{R}^n)')$-coalgebra object in E_{m-n}-algebras which proves that the iterated Bar construction given by Definition 9.15 is a functor from augmented E_m-algebras to E_n-$coAlg\Big(E_{m-n}$-$Alg^{aug}\Big)$. This functor agrees (in the $(\infty,1)$-category E_{m-n}-Alg^{aug}) with the one given in Section 9.1 by Proposition 9.14. The identification of the two E_n-coalgebras structure is done as in the proof of Proposition 7.12.

Dualizing the construction of the locally constant $N(\mathrm{Disk}(\mathbb{R}^n)')$-coalgebra structure shows that the dual $RHom(Bar^{(n)}(A), k)$ of the Bar construction has a locally constant $N(\mathrm{Disk}(\mathbb{R}^n)')$-algebra structure with the one given in Remark 7.11 and

thus with the one of Theorem 7.7. Hence, by Proposition 7.19 with the centralizer $\mathfrak{z}(A \to k)$ of the augmentation.

That the algebraic of $Bar^{(n)}(A)$ agrees with the one in [F1] follows from Dunn Theorem (see [L-HA, F1] or Theorem 2.16) once we know that $Bar^{(1)}(A)$ is equivalent, as an E_1-coalgebra to the standard Bar construction $Bar^{std}(A)$. By homotopy invariance, we may assume that A is a differential graded associative algebra. By Lemma 9.13, we have an natural equivalence $Bar(A) \cong Bar^{std}(A)$ and further the (two constructions) of the Bar construction computes the derived functor $k \otimes^{\mathbb{L}}_A k$. The coalgebra structure of $Bar^{std}(A)$ is induced by the comultiplication $\delta : Bar^{std}(A) \to Bar^{std}(A) \otimes Bar^{std}(A)$ which realized the following map of derived functors (in k-Mod$_\infty$):

$$\delta : k \otimes^{\mathbb{L}}_A k \cong k \otimes^{\mathbb{L}}_A A \otimes^{\mathbb{L}}_A k \stackrel{id\otimes^{\mathbb{L}}_A \epsilon \otimes^{\mathbb{L}}_A id}{\longrightarrow} k \otimes^{\mathbb{L}}_A k \otimes^{\mathbb{L}}_A k \cong \left(k \otimes^{\mathbb{L}}_A k\right)^{\otimes 2}. \tag{74}$$

The construction (70) can be rewritten as

$$Bar(A) \;\cong\; k \underset{A}{\overset{\mathbb{L}}{\otimes}} \int_I A \underset{A}{\overset{\mathbb{L}}{\otimes}} k$$

using the natural $A \otimes A^{op} \cong \int_{S^0} A$-module structure of $\int_I A$. Now the E_1-coalgebra structure of $Bar(A)$ is given by the inclusion of two disjoint open intervals I_1 and I_2 inside I. We denote J_1, J_2, J_3 the three disjoint intervals whose union is the complement $I \setminus (I_1 \cup I_2)$. Unfolding the definition of the map $\gamma_{I_1,I_2,I}$ given by Lemma 9.20 and using excision for factorization homology (see [L-HA, F1, GTZ2, AFT]), we find that, $\gamma_{I_1,I_2,I}$ is the composition

$$\begin{aligned} Bar(A) \;\cong\; k \underset{A}{\overset{\mathbb{L}}{\otimes}} \int_I A \underset{A}{\overset{\mathbb{L}}{\otimes}} k \longrightarrow k \underset{A}{\overset{\mathbb{L}}{\otimes}} \int_{J_1} A \underset{A}{\overset{\mathbb{L}}{\otimes}} \int_{I_1} A \underset{A}{\overset{\mathbb{L}}{\otimes}} \int_{J_2} A \underset{A}{\overset{\mathbb{L}}{\otimes}} \int_{I_2} A \underset{A}{\overset{\mathbb{L}}{\otimes}} \int_{J_3} A \underset{A}{\overset{\mathbb{L}}{\otimes}} k \\ \stackrel{id\otimes^{\mathbb{L}}_A \epsilon \otimes^{\mathbb{L}}_A id \otimes^{\mathbb{L}}_A \epsilon \otimes^{\mathbb{L}}_A id \otimes^{\mathbb{L}}_A \epsilon \otimes^{\mathbb{L}}_A id}{\longrightarrow} k \underset{A}{\overset{\mathbb{L}}{\otimes}} \int_{I_1} A \underset{A}{\overset{\mathbb{L}}{\otimes}} k \underset{A}{\overset{\mathbb{L}}{\otimes}} \int_{I_2} A \underset{A}{\overset{\mathbb{L}}{\otimes}} k \cong Bar(A) \otimes Bar(A). \end{aligned} \tag{75}$$

Hence, the underlying coproducts maps of E_1-coalgebra structure on $Bar(A)$ realize the map (74) and thus induces the E_1-coalgebra structure of $Bar^{std}(A)$ under the equivalence given by Lemma 9.13. $\square$

Remark 9.22. Note that E_1-$coAlg\big(E_1$-$Alg\big)$ is equivalent to the $(\infty, 1)$-category of bialgebras in k-Mod$_\infty$. We think of E_p-$coAlg\big(E_q$-$Alg\big)$ as analogues of bialgebras with some commutativity and cocommutativity conditions lying in between bialgebras and commutative and cocommutative bialgebras. In particular, Proposition 9.21 implies that the Bar construction of an E_2-algebra is naturally a (homotopy) bialgebra.

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